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Role of error-based activities in the development of mathematical thinking skills

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Abstract

An abstract is a brief, comprehensive summary of the contents of the article; it allows readers to survey the This study aims to examine the development of the mathematical thinking skills of pre-service elementary mathematics teachers through error-based activity practices. The study was designed as a case study, which is one of the qualitative study designs. The study group of the research consisted of five pre-service mathematics teachers studying at a Turkish university. In selecting the study group, the criterion sampling method that is one of the purposive sampling methods was employed. Data were collected through face-to-face interviews with pre-service teachers and diaries kept by them. Data were analyzed using the descriptive analysis method. Based on the findings obtained from the study, the error-based activities were observed to positively affect specifically the specialization and generalization processes of pre-service teachers. Following these two processes, the assumption and persuasion processes of the pre-service teachers were also revealed to have improved with the practice of error-based activity implementation. In conclusion, error-based activities were observed to have positively affected the mathematical thinking processes of pre-service teachers.

Keywords: Error-based activities; mathematical thinking; pre-service teachers

1. Introduction

Students of today are expected to be individuals that can solve problems, make assumptions by reasoning, and employ high-level thinking skills. This points to the importance of mathematical thinking in school education and in learning mathematics (Stacey, 2006). Indeed, mathematical thinking is a mental activity that covers reasoning, logical and critical thinking skills, and ensures success in many professions. Cuoco et al. (1996) stated that the development of mathematical thinking is associated with developing habits of mind, and this can be done by establishing new ways to describe,

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abstract, and connect situations and make predictions as well as by hypothesizing and testing them.

Mathematical thinking is the sum of certain mathematical processes rather than thinking over a single subject (Burton, 1984). In other words, in terms of mathematical thinking, it is important to understand how mathematicians make predictions to prove a theorem, rather than figuring out how they prove these theorems (Polya, 1945). Similarly, to think mathematically in case of a problem, it is necessary to examine the problem from different dimensions rather than finding its answer (Yeşildere & Türnüklü 2007). In other words, when faced with a problem, individuals should think over and analyze it. In this sense, the development of mathematical thinking is among the main objectives of mathematics teaching.

Mathematical thinking is a process that individuals use while solving problems consciously or unconsciously in their daily lives (Alkan & Bukova Güzel, 2005). It is a cognitive and affective action enabling individuals to make sense of life. This action provides for the mental and spiritual development of an individual. To support this development, errors as negative information can also be utilized in a way to complement positive information (Minsky, 1983; Eriksson & Par'iainen, 2006). In learning, errors are considered as opportunities enabling effective learning by benefiting from negative information besides positive information. In learning mathematics, errors lead the individual to get the correct information by distinguishing between the correct and false ones (Borasi, 1989; Heinze, 2005; VanLehn, 1999). Accordingly, it is argued that errors play a positive role in students' understanding of the subjects and concepts in mathematics, in problem-solving, and in developing high-level thinking skills (Borasi, 1994; Hill, Ball, & Schilling, 2008).

Errors serve as a starting point for students to start questioning in the discipline of mathematics (Schleppenbach, Flevares, & Sims, 2007; Borasi, 1994). Indeed, an individual encountering an error or difficulty in case of a problem is directed to think through self-evaluation and self-regulation. Such thinking is a mathematical thinking process that covers reasoning, relating, and connecting within the scope of problem-solving. Accordingly, in education, errors and difficulties might be regarded as learning opportunities to improve the mathematical thinking process (Steuer & Dresel, 2015; Seifried & Wuttke, 2010; Borasi, 1994, Gedik & Konyalıoğlu, 2019). Formed by benefiting from the errors of students as a method with a positive effect in teaching, error-based activities are mentioned as a teaching activity in the literature (Billi, Özkaya, Çiltaş, & Konyalıoğlu, 2021). Teachers' way of practicing error-based activities in their classrooms shapes the experiences of students by revealing appropriate ways of addressing errors and preferred attitudes towards these. The effects of these can be observed all through the lives of students and shape their perception of errors and way of dealing with them (Santagata, 2005).

Literature covers separate studies on the importance of error-based activities and mathematical thinking in mathematics education. However, there is no study relating error-based activities with mathematical thinking to address how these activities lead to a change in the thinking processes of pre-service teachers. This study was carried out assuming that, by their nature, error-based activities might improve the mathematical thinking processes of pre-service teachers.

Research questions sought to be answered in this study are as follows.

- How did the practice of error-based activities contribute to the development of pre-service teachers' mathematical thinking?
- What are the opinions of pre-service teachers on the development of their mathematical thinking?

Theoretical Framework

Mathematical Thinking

In the literature, there are different definitions of mathematical thinking. While some researchers relate mathematical thinking with the development of mathematical concepts (Tall, 1995). Some researchers associated it with processes such as estimation, generalization, and verification that can be used in problem-solving. Features distinguishing mathematical thinking from other ways of thinking and rendering it mathematical are the operations, processes, and dynamics (Burton, 1984; Mason, Burton, & Stacey, 2010; Polya, 1945; Schoenfeld, 1992). Mason, Burton, and Stacey (2010) expressed the basic processes of mathematical thinking as specialization, generalization, making assumptions, and persuasion. Features of these processes are as follows:

2. Theoretical Framework

2.1. Mathematical Thinking

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Table 1. Processes of Mathematical Thinking

Process	Features
Specialization	It involves working on specific case studies to examine and better understand the given problem.
Generalization	Based on the sample cases examined, it seeks to clarify the intuitions of some basic relations to reveal the pattern of the problem.
Assumption	It is the process of reaching a judgment by examining the relations in case studies.
Verification	Finding out and communicating the reason why something is correct. It is the indication of whether the assumption is correct or false.

3. Method

3.1. Study design

This study aims to examine the development of the mathematical thinking skills of pre-service elementary mathematics teachers through error-based activity practices. Accordingly, the research was designed as a case study. A case study is an approach that involves an in-depth examination of a limited system using multiple data collection to gather systematic information about how it works and functions (Chmiliar, 2010). In this study, an environment was established to enable five pre-service teachers to make error-based activities every week. Following the activities conducted each week, questions to be answered in the diaries were provided. Participants answered these questions regularly every week. At the end of this regular practice, semi-structured interviews were conducted with the pre-service teachers. This lasted three weeks. Each week, three error-based activity questions were shared and discussed with the participants. Each of the practices lasted nearly two hours.

3.2. Study group

The study was carried out with five third-grade pre-service teachers studying in the elementary mathematics education program of a state university. In selecting the study group, the criterion sampling method that is one of the purposive sampling methods was employed. In determining the participants, criteria of being "voluntary" and at "different levels of success" were considered. According to the grade point averages of the first semester of the third year; two students were selected from the 3.5–3.0 grade range, two from the 3–2.5 grade range, and 1 from the 2.5–2 grade range.

3.3. Data collection tools

Data collection tools consist of diaries requested from the participants after each practice and semi-structured interviews made with each of them. In each practice carried out with the pre-service teachers for three weeks, three error-based activity questions were discussed. Each week when error-based activities are practiced, one question each on algebra, geometry, and proof was given. These questions were taken from the book titled "Error-Based Activities in Mathematics Teaching" written by Konyalıoğlu, Özkaya, and Gedik (2019). Questions prepared for the face-to-face interviews with the pre-service teachers to be asked at the end of the practice are as follows:

1. Did the practice cause a change in your perspective on questions and ways of the solution? If any, what kind of change did you observe in yourself?
2. What do you think about whether this practice has led you to ways of thinking different from your previous thinking habits?

3.4. Data analysis

In the data analysis, the conceptual framework of mathematical thinking processes developed by Mason, Burton, and Stacey (2010) was taken as the basis. Data obtained from the diaries kept by the pre-service teachers were coded based on four processes under this framework. Findings of each three weeks of the practice process carried out with the pre-service teachers were presented in a table together with the necessary explanations to ensure a better understanding. The findings obtained were supported by direct quotations from the explanations written by the pre-service teachers in their diaries. Whilst data obtained from face-to-face interviews were presented with direct quotations, by making the necessary explanations.

4. Findings

Findings obtained as a result of this practice with pre-service teachers were presented separately to display the change in each pre-service teacher following the three-week practice period by considering the mathematical thinking processes specified in the theoretical framework section. Findings of each week obtained from each pre-service teacher were presented in a table within the framework of the processes. Tables were prepared to identify what kind of changes in which processes are observed in the explanations of the pre-service teachers for the questions related to the practice. For three weeks, error-based activity questions were solved with the pre-service teachers. Answers given by T1 by the processes are given below in the tables.

Table 2. Findings obtained from the Pre-Service Teacher T_1 by Weeks:

	Week 1			**Week 2**			**Week 3**		
Questions / **Components**	Q 1	Q 2	Q 3	Q 1	Q 2	Q 3	Q 1	Q 2	Q 3
Specialization	X	X	X	X	X	X	X	X	X
Generalization	X	X	X	X	X	X	X	X	X
Making Assumptions			X	X		X		X	
Persuasion				X		X		X	

Table 2 shows that the pre-service teacher applied the formula directly in the first and second practice questions of the error-based activity and obtained a wrong result in the first week. Therefore, the answers were addressed in the dimension of specialization and generalization processes of mathematical thinking. In the third question of the first week, the pre-service teacher regarded the proof to be demonstrated as an assumption and concluded that the initial theorem was correct since the last equation was found out to be correct. In this question, the pre-service teacher provided false proof by making a false assumption. In the first of the questions asked in the second week, the pre-service teacher obtained a wrong solution by applying a formula. In the second question, the teacher looked at the angle-side relation and stated that the question was wrong. In the last question, the pre-service teacher noticed the erroneous parts in the proof given and tried to reach the required result by making assumptions. In the last week of the practice, the pre-service teacher reached the wrong result with the formula they were familiar with in the first question. In the second question, they noticed the errors and reached the correct result. In the third question, the pre-service teacher obtained the correct result by testing with numbers among the options given. Therefore, they were addressed in the dimension of specialization and generalization processes in mathematical thinking.

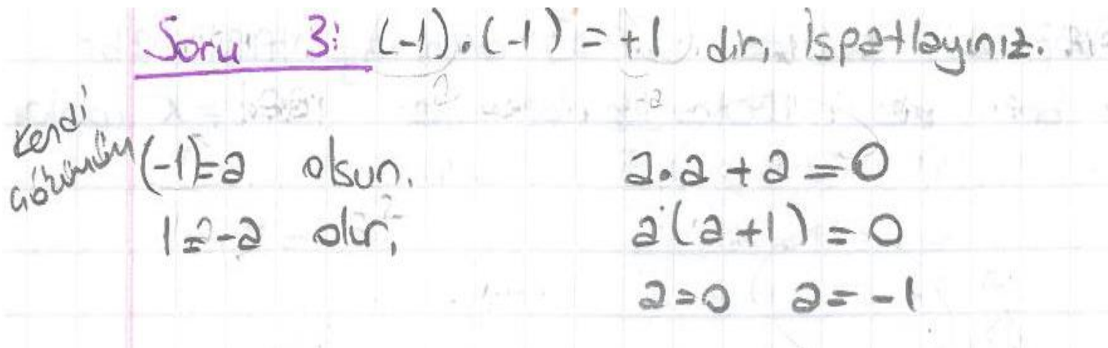


Figure 1. T$_1$'s Answer to Question 3 in Week 1

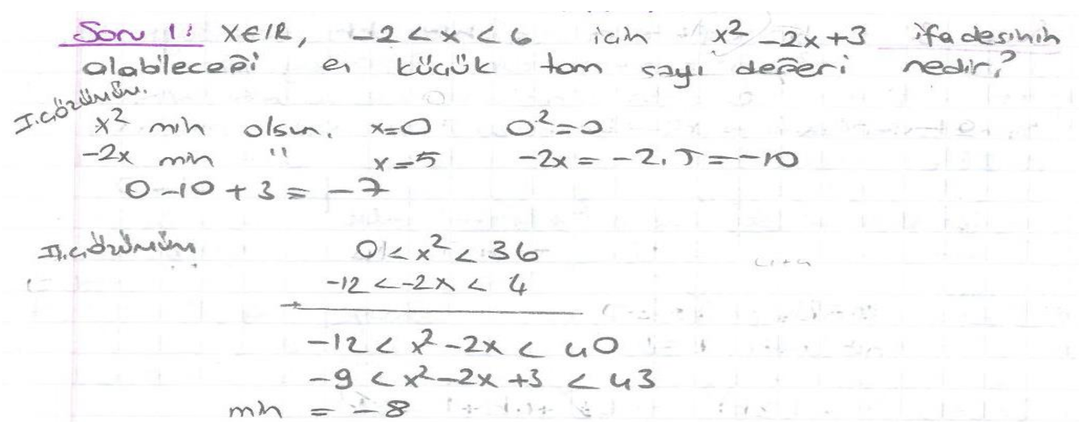


Figure2. T$_1$'s Answer to Question 2 in Week 2

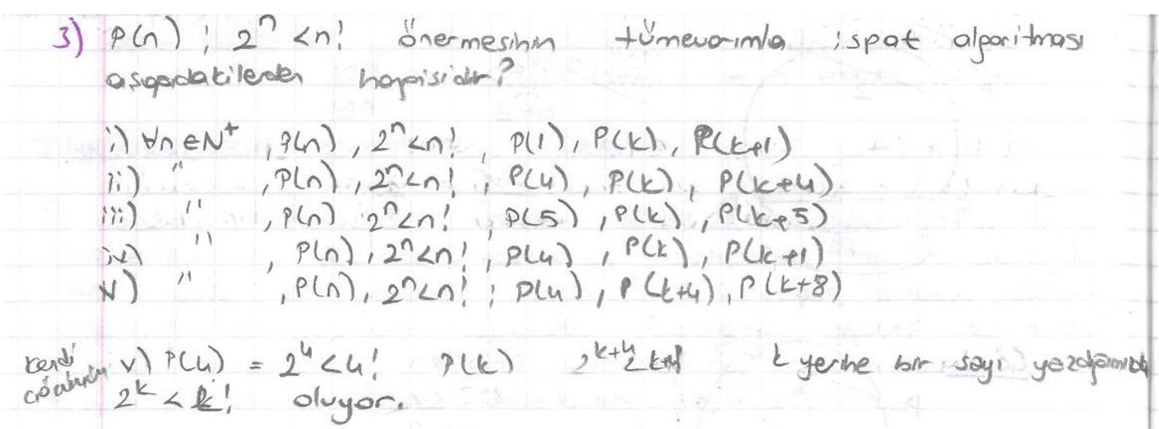


Figure 3. T$_1$'s Answer to Question 3 in Week 3

Table 3. Findings obtained from the Pre-Service Teacher T_2 by Weeks:

	Week 1			**Week 2**			**Week 3**		
Questions / **Components**	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3
Specialization	X	X	X	X	X	X	X	X	
Generalisation	X	X	X	X	X	X	X	X	
Making Assumptions	X			X	X			X	
Persuasion	X			X	X			X	

Table 3 displays that the pre-service teacher noticed the cause of the error in the first practice question of the error-based activity in the first week and reached the right solution. In this sense, she applied all processes of mathematical thinking. In the second question, she did not realize the error and reached the wrong solution by applying a formula. In Question 3, she stated that it was a theorem without providing proof. She tried to generalize this by giving different examples. Realizing that there is a contradiction in the first and second questions asked in the second week, she indicated the cause of the error. In the third question, assuming that the given statement was correct she stated that the statement obtained was also correct. She could not realize the existence of an error in this question. In the first question given in the last week, she solved it by applying the formula. Therefore, she used specialization and generalization among mathematical thinking processes. In the second question, she realized the error in the solution and reached the correct solution themselves. Thus, she applied all mathematical thinking processes. In the last question, she stated that they did not know the inductive method and could not make any comments, so this question was not evaluated in the table.

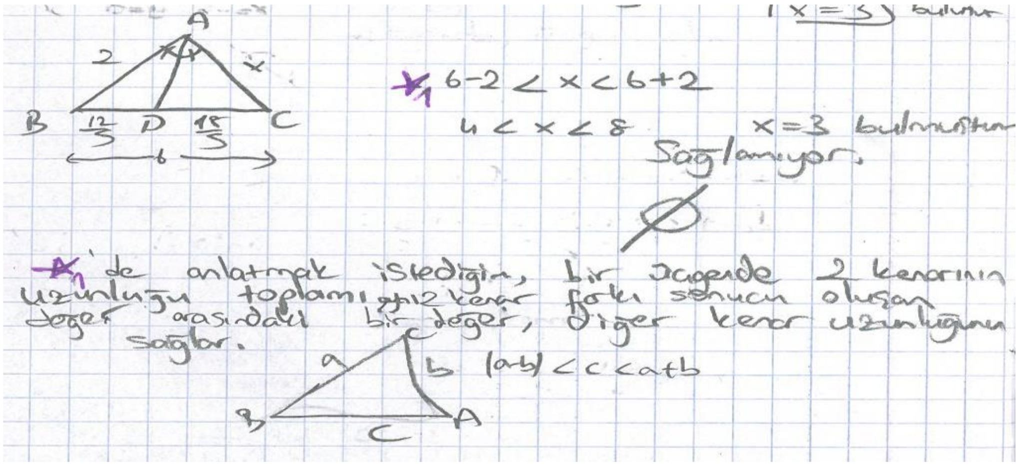


Figure 4. T₂'s Answer to Question 1 in Week 1

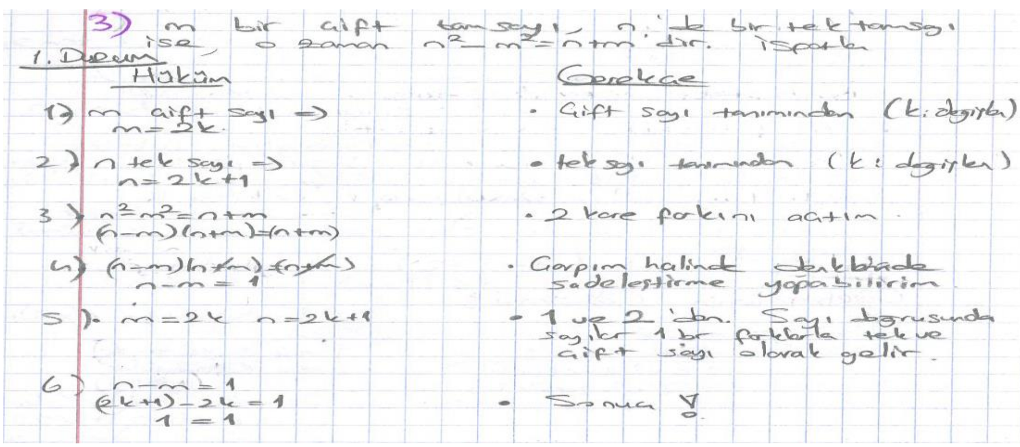


Figure 5. T₂'s Answer to Question 3 in Week 2

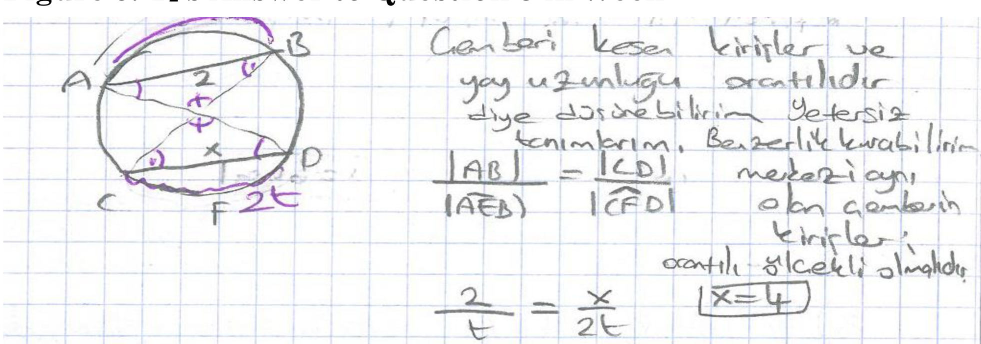


Figure 6. T₂'s Answer to Question 1 in Week 3

Table 4. Findings obtained from the Pre-Service Teacher T_3 by Weeks:

	Week 1			**Week 2**			**Week 3**		
Questions / **Components**	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3
Specialization	X	X	X	X	X	X	X	X	X
Generalization	X	X	X	X	X	X	X	X	X
Making Assumptions	X	X	X	X		X		X	
Persuasion	X	X		X		X		X	

Table 4 shows that the pre-service teacher noticed the errors in the first and second practice questions of the error-based activity and reached the correct result in the first week. In this sense, they applied all the processes of mathematical thinking. In the third question of the first week, the pre-service teacher regarded the proof to be demonstrated as an assumption and concluded that the initial theorem was correct since the last equation was found to be correct. In this question, the pre-service teacher provided false proof by making a false assumption. Accordingly, among the mathematical thinking processes, specialization, generalization, and making assumptions processes were practiced. They noticed the error in the first question of the second week and reached the correct result. They applied all processes of mathematical thinking. In the second question, they applied two different solutions and stated the smaller of these as correct. The pre-service teacher reached the correct solution in this question but considered that the first solution was correct as she had a smaller result in the first solution. In the third question, they realized the error and stated the deficient part in the question. Thinking that they understood the first question of the third week of the practice, she applied the formula directly and obtained a wrong result. Therefore, the answers were addressed in the dimension of specialization and generalization processes of mathematical thinking. They realized the error in the second question and developed the right solution. In the third question, she tried to reach the correct result by assigning a value to the statements in the options.

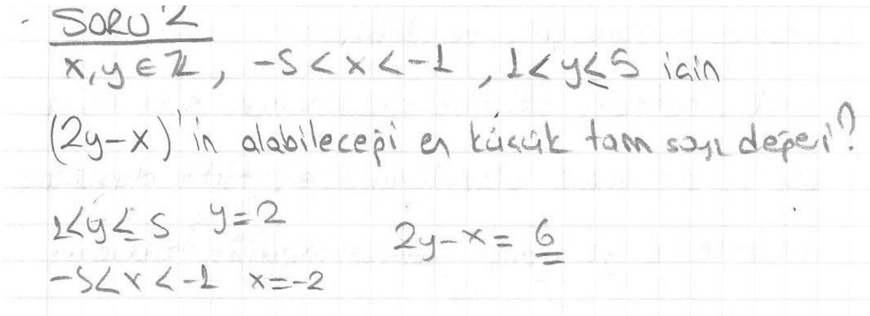


Figure 7. T3's Answer to Question 2 in Week 1

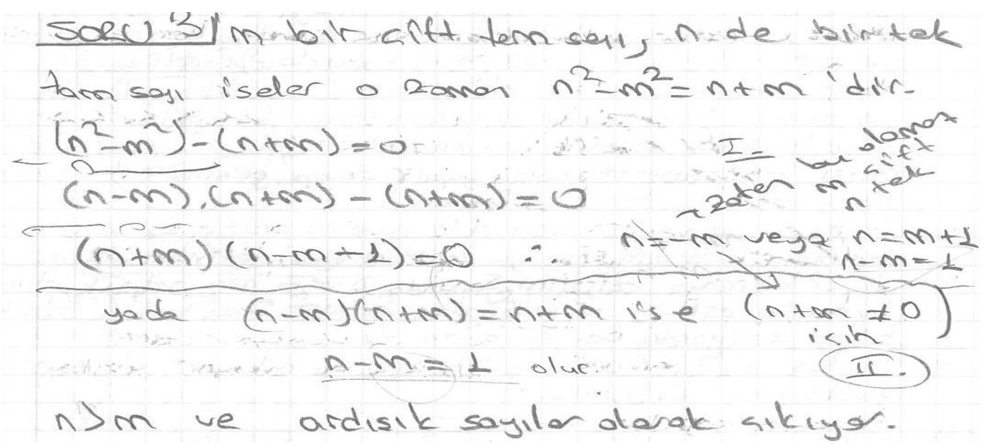


Figure 8. T3's Answer to Question 3 in Week 2

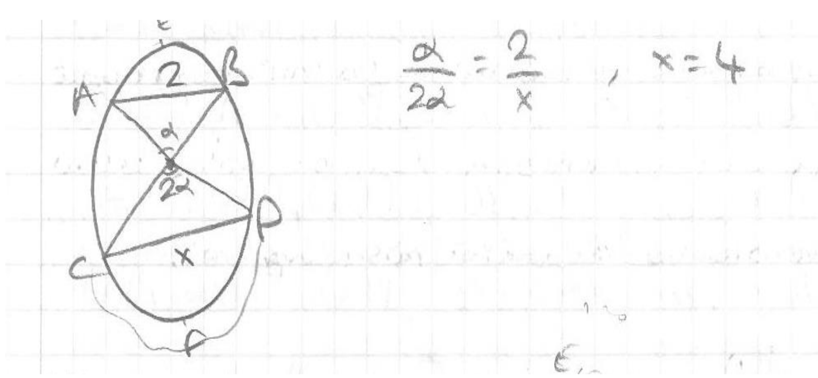


Figure 9. T3's Answer to Question 1 in Week 3

Table 5. Findings obtained from the Pre-Service Teacher T_4 by Weeks:

	Week 1			**Week 2**			**Week 3**		
Questions / **Components**	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3
Specialization	X	X	X	X	X	X	X	X	X
Generalization	X	X	X	X	X	X	X	X	X
Making Assumptions			X	X					
Persuasion				X					

Table 6 displays that the pre-service teacher applied the formula directly in the first and second practice questions of the error-based activity and obtained a wrong result in the first week. Therefore, the answers were addressed in the dimension of specialization and generalization processes of mathematical thinking. In the third question of the first week, the pre-service teacher regarded the proof to be demonstrated as an assumption and concluded that the initial theorem was correct since the last equation was found out to be correct. In this question, the pre-service teacher provided false proof by making a false assumption. They noticed the error in the first question of the second week and reached the correct result. They applied all processes of mathematical thinking. They reached the wrong solution by adding the left-hand and the right-hand sides, in the second question. They could not notice the error made. In the last question, they could not realize the deficiency in the question. They provided the proof by supposing that the two numbers are successive. In the last week of the practice, they could not notice the errors in the first and second questions and reached the wrong result. They reached the correct result by testing with numbers among the options of the third question. Therefore, they were addressed in the dimension of specialization and generalization processes in mathematical thinking.

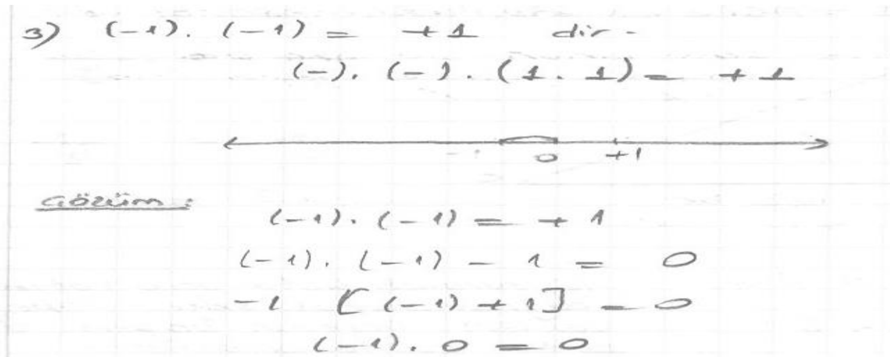


Figure 10. T$_4$'s Answer to Question 3 in Week 1

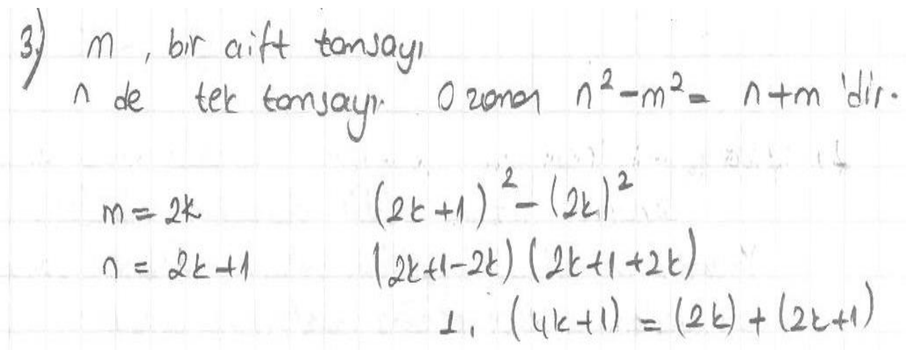


Figure 11. T$_4$'s Answer to Question 3 in Week 2

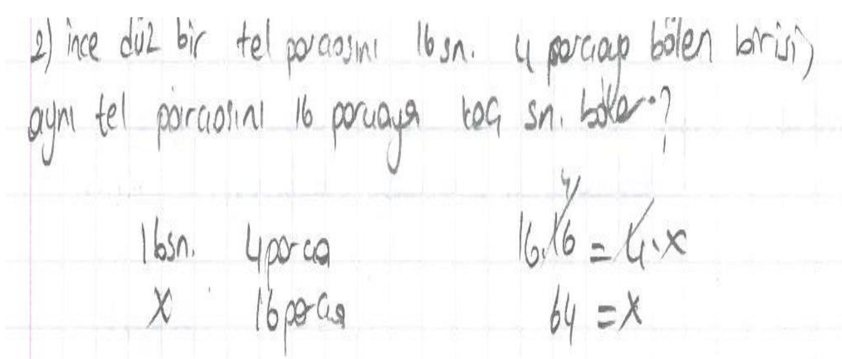
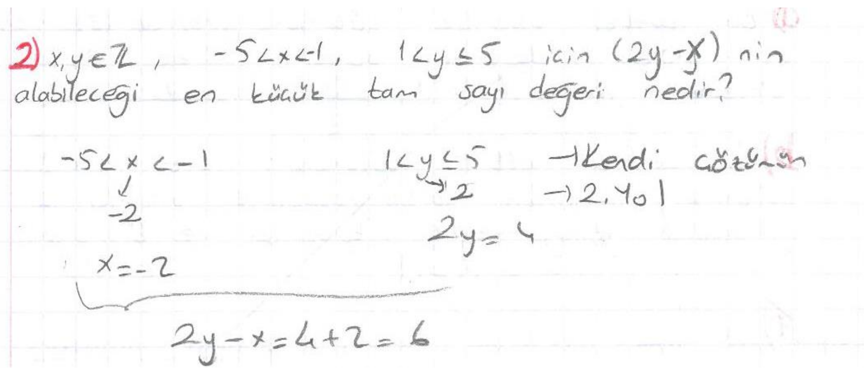


Figure 12. T$_4$'s Answer to Question 2 in Week 3

Table 6. Findings obtained from the Pre-Service Teacher T_5 by Weeks:

	Week 1			**Week 2**			**Week 3**		
Questions / **Components**	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3
Specialization	X	X	X	X	X	X	X	X	X
Generalization	X	X	X	X	X	X	X	X	X
Making Assumptions		X	X	X					
Persuasion		X		X					

Table 7 shows that the pre-service teacher applied the formula directly in the first practice question of the error-based activity and obtained a wrong result in the first week. Therefore, the answers were addressed in the dimension of specialization and generalization processes in mathematical thinking. In the second question of the first week, the pre-service teacher realized the error and reached the right solution by considering the domain set. In the third question, the pre-service teacher regarded the proof to be demonstrated as an assumption and concluded that the initial theorem was correct since the last equation was found out to be correct. In this question, the pre-service teacher provided false proof by making a false assumption. They noticed the error in the first question of the second week and reached the correct result. They applied all processes of mathematical thinking. They reached the wrong solution by adding the left-hand and the right-hand sides in the second question. They could not notice the error made. In the last question, they could not realize the deficiency in the question. They provided the proof by supposing that the two numbers are successive. In the last week of the practice, they could not notice the errors in the first and second questions and obtained the wrong result. They obtained the correct result by testing with numbers among the options of the third question. Therefore, they were addressed in the dimension of specialization and generalization processes in mathematical thinking.


2) $x, y \in \mathbb{Z}$, $-5 < x < -1$, $1 < y \leq 5$ için $(2y - x)$ nin alabileceği en küçük tam sayı değeri nedir?

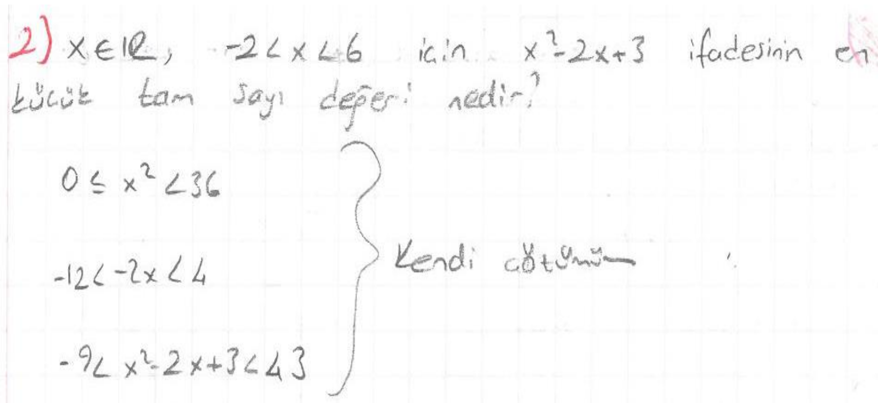
$-5 < x < -1$ → -2 ; $1 < y \leq 5$ → 2 → Kendi çözümüm → 2. Yol

$2y = 4$

$x = -2$

$2y - x = 4 + 2 = 6$

Figure 13. T$_5$'s Answer to Question 2 in Week 1

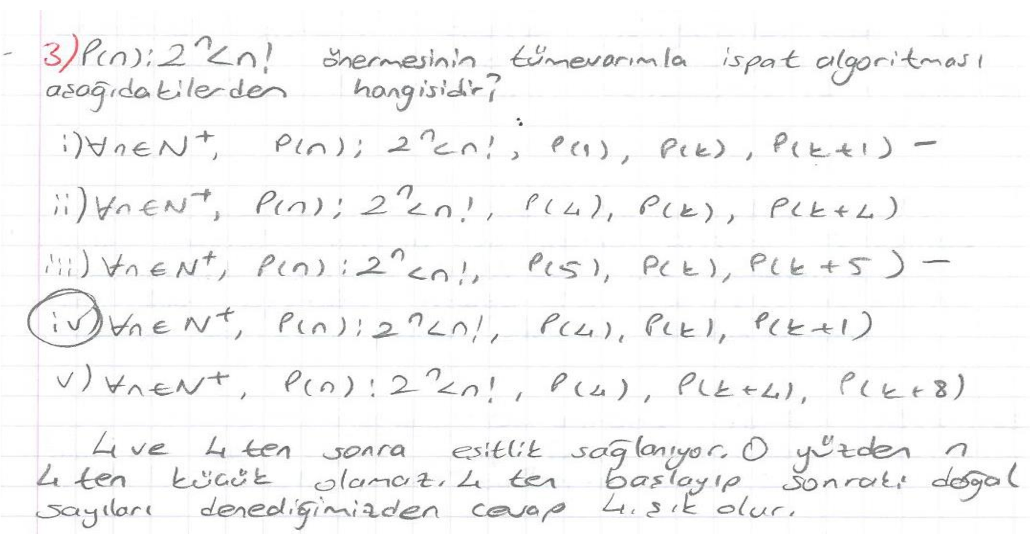

2) $x \in \mathbb{R}$, $-2 < x < 6$ için $x^2 - 2x + 3$ ifadesinin en küçük tam sayı değeri nedir?

$0 \leq x^2 < 36$

$-12 < -2x < 4$ Kendi çözümüm

$-9 < x^2 - 2x + 3 < 43$

Figure 14. T$_5$'s Answer to Question 2 in Week 2


3) $P(n): 2^n < n!$ önermesinin tümevarımla ispat algoritması aşağıdakilerden hangisidir?

i) $\forall n \in N^+$, $P(n): 2^n < n!$, $P(1)$, $P(k)$, $P(k+1)$

ii) $\forall n \in N^+$, $P(n): 2^n < n!$, $P(4)$, $P(k)$, $P(k+4)$

iii) $\forall n \in N^+$, $P(n): 2^n < n!$, $P(5)$, $P(k)$, $P(k+5)$

iv) $\forall n \in N^+$, $P(n): 2^n < n!$, $P(4)$, $P(k)$, $P(k+1)$

v) $\forall n \in N^+$, $P(n): 2^n < n!$, $P(4)$, $P(k+4)$, $P(k+8)$

4 ve 4 ten sonra eşitlik sağlanıyor. O yüzden n 4 ten küçük olamaz. 4 ten başlayıp sonraki doğal sayıları denediğimizden cevap 4. şık olur.

Figure 15. T$_5$'s Answer to Question 3 in Week 3

Findings of the face-to-face interview:

In the face-to-face interviews, the pre-service teachers were asked to answer the following questions.

1. Did the practice cause a change in your perspective on questions and ways of a solution? If any, what kind of change did you observe in yourself?

2. What do you think about whether this practice has led you to ways of thinking different from your previous thinking habits?

Concerning the first question asked in the face-to-face interview, the pre-service teachers stated that the questions in the practice were easy for them and that they felt familiar with the questions at the outset. Regarding this question, the pre-service teachers noted that in the first week of the practice, they did not think about the root of the question and the alternative ways of thinking. They noted that they started to examine the root of the question to look for the errors in the root of the question, to look for the errors in the ways of solution, to approach questions with suspicion, to try alternative solutions, to read the question carefully and in an in-depth manner, to notice the integrity of the question with the solution, and to notice the connections. Direct quotations from the opinions of the pre-service teachers that participated in the practice are given below.

"I was not used to examining the question. Later, I began to get to the root of the question and examine it in more detail" (T_1)

"At the beginning, I used to focus directly on the solution. At the end of the practice, I began to approach the questions and solutions with suspicion" (T_2)

"Previously, I used to think simply. At the end of the process, I saw that a question covers several subjects" (T_3)

"At the beginning, I used to focus directly on the solution. At the end of the practice, I began to look at the questions in more detail and with a broader perspective" (T_4)

"Initially, I used to assume that the question was familiar and began to solve it directly. At the end of the practice, I learned to read the root of the question and what is asked" (T_5)

Concerning the second question asked in the interview, the pre-service teachers stated that they used to be operation-oriented, not method-oriented, that they referred to memorization mentality; but at the end of the practice, they noted that they learned to explore what was being asked in the questions, started to look into the questions in-depth, questioned, and sought alternative ways of solution. Direct quotations from the opinions of the pre-service teachers that participated in the practice are given below.

"At the start of the practice, I was operation-oriented. Now, I try to solve the questions using different ways." (T_1)

"After the practice, I try to understand what is asked in the questions. I observed that those referring to memorization could not reach the correct answer to the questions and I began to apply different ways of thinking." (T_2)

"I began to think differently. Now, I approach more skeptically." (T_3)

I began to look at the questions in more detail and with a broader perspective. I check more often whether my solution is correct or not." (T_4)

"Previously, I used to be method oriented. Now, I think more thoroughly, and with a broader perspective." (T_5)

5. Discussion

This study aims to reveal the development of mathematical thinking by performing error-based activities on pre-service elementary mathematics teachers. In line with the determined aim, a three-week practice on error-based activities was carried out. Each week, three error-based questions were asked and discussed with the pre-service teachers. Face-to-face interviews were held with the pre-service teachers to take their opinions thoroughly on the relevant practice. In line with the determined theoretical framework, statements of the pre-service teachers were coded and evaluated according to the mathematical thinking processes.

In the questions presented in the first week of the study, the candidates were observed to mostly focus on the specialization and generalization processes. In the first and second questions of the first week, two of the pre-service teachers questioned the root of the question and understood the problem. Therefore, they conducted verifications by making assumptions regarding the question. Three of the pre-service teachers were established to not question the root of the question, did not try to understand the problem, and solved it by applying the formulas and theorems they already know. After reaching a solution, they did not make any assumptions and conducted verifications. In the third question, four of the pre-service teachers generalized, thinking that they had understood the question but made wrong assumptions about the correct solution. Therefore, they could not conduct verifications.

In the questions presented in the second week, the pre-service teachers generally started to question the root of the question. In the first question of the second practice, they resorted to specialization by showing the angles on the figure with letters and generalized by trying to correlate the angles and the sides. All the pre-service teachers noticed the discrepancy between the angle and the side and made assumptions. They tested and verified their assumptions, stating that the root of the question was wrong. In the inequality question of the practice, only one pre-service teacher questioned the root of the problem, reached a solution by making the correct assumption, and realized the verification process. The other four pre-service teachers made an incorrect generalization by trying to find a relation between the given interval and the given statement. Therefore, they realized the specialization and generalization processes of mathematical thinking. In this question, all the pre-service teachers paid attention to the set examined. In the last question of the practice, three of the pre-service teachers did not question the root of the question and made an implicit assumption that m and n were successive. Pre-service teachers studied the proof by generalizing. Three of the pre-service teachers tried to correlate the numbers, and two of them solved it by turning the statement into an equation. Thus, all of them resorted to generalization. Pre-service teachers solving the equation concluded that m and n are successive and realized the mathematical thinking processes. However, other pre-service teachers, resorted to proving based on their first theorem but did not mention the error at the root of the question. Therefore, they were able to realize specialization and generalization dimensions of mathematical thinking processes.

In the practice carried out in the third week, the pre-service teachers were observed to have remained in the dimension of specialization and generalization of the mathematical thinking processes. In the first question of the practice, the pre-service teachers applied the formula they knew by rote without examining the question in detail and reached the wrong result. In the second question, three pre-service teachers noticed that dividing into four parts means three cuts and applied the correct solution. On the other hand, the other two pre-service teachers did not think this and reached the wrong solution by applying the formula of proportion. In the last question of the practice, four pre-service teachers reached the correct result by assigning values. Therefore, they did not make any assumptions or verification. One of the pre-service teachers stated that they do not know the inductive method.

As a result of this study, it was determined that the pre-service teachers focused more on the specialization and generalization processes. In the first week of the practice, assumption and verification among the other processes were not in focus that much. In the practice, the pre-service teachers were generally observed to have solved by applying a formula based on their prior knowledge after examining special values and special shapes or without following any of these steps. Arslan & Yıldız (2010) and Steele & Johanning (2004) also encountered a similar situation and determined that students realizing the correlations between problems applied previous generalizations on new problems without studying special cases. During the research process, all pre-service teachers were observed to have rechecked the questions with the rule to verify their generalizations and that they were able to express how they reached the generalization. Pre-service teachers were able to apply specialization and generalization processes on the questions easily as they encountered more questions requiring operational skills rather than conceptual skills, in their education life (Arslan & Yıldız, 2010; Keskin et al. 2013; Yıldız, 2016). This is in line with the studies of Arslan & Yıldız (2010) and Arslan (2016). In the second and third weeks of the practice, some of the pre-service teachers were observed to have focused on the assumption and verification processes while some could not. It was discovered that the pre-service teachers who could not realize assumption and verification among the thinking processes did not think much over the questions they could not solve or could not reach the correct answer and did not insistently deal with the questions they found difficult. However, the pre-service teachers realizing the assumption and verification processes thought over the questions and noticed the erroneous and deficient parts, and generally made the right assumptions and realized the verification process. Similarly, in the study of Uzun, Topan, Demir, and Çelik (2019), the pre-service teachers could realize specialization and generalization among mathematical thinking processes while they rarely resorted to assumptions and could not use verification.

6. Conclusions

This study revealed that dealing with error-based activities improves mathematical thinking processes of pre-service teachers such as specialization, generalization, making assumptions, and verification. In the interviews, pre-service teachers stated that they were puzzled before the questions. They noted that they were familiar with the questions but did not notice the errors and deficiencies in the questions. They could not notice the errors and deficiencies in the questions given in the first week of the study. In the following weeks, pre-service teachers began to think in-depth, in detail, analytically, and flexibly, and to examine the errors and deficiencies in the given questions more carefully. However, according to their answers to the questions, their mathematical thinking processes remained in the dimension of specialization and generalization. However, interviews pointed out that error-based activities enabled the pre-service teachers to realize that they need to benefit from assumption and verification processes as well as specialization and generalization. This suggests that error-based activities are important for understanding the deficiencies of the concepts learned and for the development of mathematical thinking processes. It is necessary to train qualified teachers that have internalized and performed mathematical thinking required in all areas of daily life and emphasized in the curriculum. Carrying out similar practices with more and varied study groups and for a longer time is considered to be beneficial for the understanding and improvement of mathematical thinking processes.

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