Available online at ijci.wcci-international.org

International Journal of Curriculum and Instruction 12(2) (2020) 191-208

IJCI
International Journal of Curriculum and Instruction

Forms of knowledge and knowing in mathematics education: Informed by concept study based on complexity theory

Emmanuel Deogratias [a] *

[a] *Doctor of Education in Mathematics Education, University of Dodoma, Dodoma 41218, Tanzania.*

Abstract

This paper starts by addressing a complexity theory and then distinguishing complex systems from complicated systems. After that, the paper addresses a concept study by connecting it with complex systems because a class is considered as a complex system having many students with different constraints, experiences, and perspectives including learning abilities, genders, and understandings. This connection may explain the need for these structures in mathematics education for individual and collective understandings.

Keywords: Complexity theory; concept study; systems of complexity theory; mathematics education

1. Introduction

1.1. Context

This paper describes the systems of complexity theory by connecting it with concept study, a model for teachers' professional development of mathematics for teaching (Davis & Renert, 2014). Then, the paper illustrates concept study by connecting it with the systems of complexity theory as a way of knowing and gaining knowledge in mathematics education. This connection is important for individual and collective understandings of the mathematical concepts in a class. After that, the paper addresses knowledge and knowing of what to discuss through concept study under the lens of complexity theory to elaborate individual teachers' and collective understandings of a mathematical concept. Later on, the paper describes the advantages of conducting concept study under the lens of complexity theory in a mathematics class. The paper ends by providing concluding

* Emmanuel Deogratias. Tel.: +255713293936
E-mail address: emmanuel.deogratias@udom.ac.tz/edeo1982@yahoo.com/deograti@ualberta.ca

thoughts about concept study under the lens of complexity theory in mathematics education.

1.2. Description of complexity theory by relating it to mathematics education

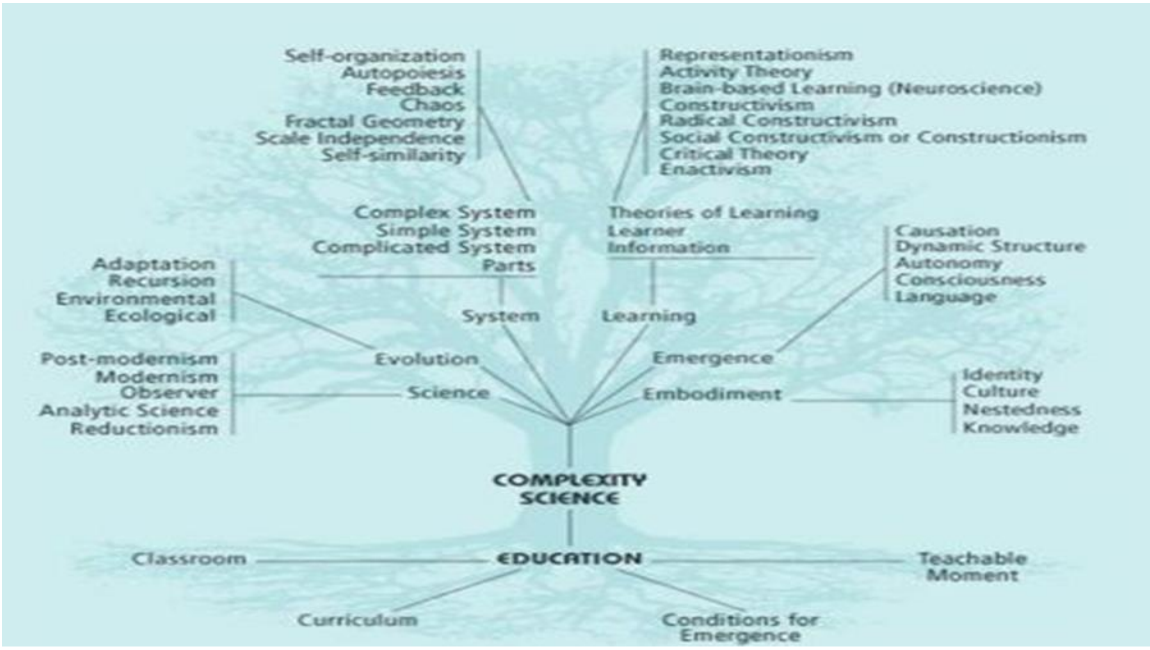
Given that concept study emerged from complexity theory, it is important to begin by gaining a foundational understanding of the theory. It has been foremost among the forms of curriculum inquiries, not only in the field of science but also in mathematics, sociology, psychology and philosophy. This theory came into the teaching and learning process due to the shift from pedagogy in which the information id transmitted from point A to point B, to pedagogy whereby point A and point B are interconnected, interrelated and interdependent as a distributed network (Davis, 2004). Complexity theory emphasizes that knowledge is dynamic and emergent allowing shared participation in a collective classroom (Davis, 2004; Newell, 2008; Davis & Sumara, 2006; Davis & Simmt, 2006).

Complexity theory is among the modes of curriculum inquiries that give both learners and teachers power in the teaching and learning process. Ideas from every individual are respected, and everyone is open minded to listen to each other and articulate the ideas into meaningful learning. This social collective learning is the major key when using this mode of curriculum inquiry in the classroom. Exploring complexity theory in mathematics is important, as the model is becoming useful to mathematics teachers and mathematics educators in the classroom. As well, it is useful in research to gain a deep insight into the area of mathematics for teaching. Mathematics for teaching is:

> a way of being with mathematics knowledge that enables a teacher to structure learning situations, interpret student actions mindfully, and respond flexibly, in
>
> ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice(Davis & Renert, 2014, p. 4).

As a teacher pursues mathematics for teaching, the classroom under the guidance of the teacher becomes socially interconnected, interrelated and interdependent, which in turn might bring the emergence of ideas and self-organization among participants in the teaching and learning environment.

In explaining these facts in mathematics education, it is important to know how complexity theory emerged. The theory emerged from science when scientists were investigating the behavior of different objects under study. The studied behavior of objects was not predictable; as a result, scientists could come up with unexpected results/emergence. Thus, I can define complexity science by opting for the definition of Newell (2008) on complexity theory as the, "way of investigating and discussing a class of phenomena from many different disciplines that is resistant to being understood through reductionist analysis" (p. 5). The phenomena and the complex objects are incompletely understood, and might contribute to the emergence and self-organization of mathematics ideas. The details of complexity science are illustrated in the figure below.



Retrieved from http://etec.ctlt.ubc.ca/510wiki/Complexity_Theory

2. Complex and complicated systems

It is important to define a "complex system", and to distinguish the term from "complicated system". In a complex system, the object is not fixed, the interactions of components are not fixed, they are non-linear, and they are not clearly defined; and the components of the complex system are subjected to co-adaptation allowing emergence of patterns. By contrast, a complicated system has many parts which are fixed, and the relationship between those parts are clearly defined as we can see in this video link: (http://www.bing.com/videos/search?q=chaos+and+complexity+theories+you+tube&FORM=VIRE7#view=detail&mid=5A5C7D0EEDB3C7202F065A5C7D0EEDB3C7202F06)

Examples of complicated systems include a clock and motor vehicle, while all living organisms are an example of a complex system. Therefore, a complicated system is mechanical while a complex system is associated with the principle of biology and evolution. Davis and Sumara (2006) give a clear distinction of complex system and complicated system as outlined below:

> …although a complicated system might have many components, the relationship among those parts is fixed and clearly defined. If it were carefully dismantled and reassembled, the system would work in exactly the same way. However, there exists some forms that cannot be dismantled and reassembled, whose characters

> are destroyed when the relationships among components are broken. Within these sorts of complex systems, interactions of components are not fixed and clearly defined, but are subject to on-going co-adaptations (Ibid, p. 11).

In a school context, the class is a complex system, consisting of different students having different learning attributes such as culture, gender, language, ways of knowing and expressing ideas/knowledge, experiences among class members, variety of education backgrounds and so on. Thus, in the teaching and learning process, listening to everyone in the learning is the key factor for every individual to be able to articulate the ideas of others, and appreciate the individual differences. Listening to one another is a complicated phenomenon as it involves applying three associated powers of listening to people in a conversation with time and space to avoid a continuous conversation and flowing of ideas. The three powers of listening that include, "evaluative, interpretive and hermeneutic" (Davis & Renert, p. 84). Teachers use these powers of listening to engage students and their mathematics learning in the classroom context. Hence, Davis and Renert (2014) call this kind of listening "three co-implicated modes of listening" (p. 86). Evaluative listening involves seeking answers, evaluating correctness, and orienting to established knowledge. Interpretive listening involves seeking interpretations, making sense of understanding, and orienting to where the learner is in the moment. And hermeneutic listening involves seeking variation, participating in meaning making, and orienting by not yet articulated possibilities. Through listening to each other in the classroom context in a complex system, learning might thus become emergent, self-organized, self-maintained and transformed (Doll, 1993, p. 57).

In addition to careful listening, it is important to think in a complex way. EDSE 610 has raised my understanding of different modes of curriculum inquiries including the complexity theory. I realize that the theory emphasizes the complexity thinking among participants. As participants we need to know, what does complexity thinking mean? To what extent do we think in a complex system? Davis and Sumara (2006) define complexity thinking as, "a way of thinking and acting" (p. 18). Using Davis and Sumara's (2006) definition of complexity thinking, I argue that people think and act differently on a certain concept or idea, so it is important to appreciate the individual differences, and value the contributions of, every individual learner, to avoid rupture that might occur during the teaching and learning process. Also, I am continuing to keep in mind that the complexity theory emphasizes social collectives rather than individuals in the learning enterprise.

3. Concept study

3.1. Meaning of a concept study

Concept study is "a participatory methodology through which teachers interrogate and elaborate their mathematics" (Davis & Renert, 2014, p. 35). It involves concept analysis and lesson study (Davis & Renert, 2014). Concept analysis in a concept study focuses on activities that, "teachers engage in to improve the quality of their teaching and enrich students' learning experiences" (Fernandez & Yoshida, 2004, p. 4). In additional, Usiskin et al. (2003) state that a concept analysis

> involves tracing the origins and applications of a concept, looking at the different ways in which it appears both within and outside mathematics, and examining the various representations and definitions used to describe it and their consequences (p. 1)

Lesson study is part of the concept study, sometimes it is used as an independent approach in teaching and learning. Lesson study is "oriented towards new pedagogical possibilities through participatory, collective, and ongoing engagement" (Davis & Renert, 2014, p. 39). Concept study is more focused on the actual mathematics content of teaching through allowing teachers as a community of experts to share ideas on a certain concept in a collective way. Including a diversity of participants, concept study creates a complex system that might open up emergent possibilities.

3.2. Pioneers of concept study in mathematics education

Concept study can be seen in the work of mathematics educators, including Davis and Renert (2014), Davis and Simmt (2006), Davis and Sumara (2006), and Davis (2009). For instance, Davis and Simmt (2006) conducted a concept study on multiplication with in-service mathematics teachers and educators using complexity science as a framework of interpretation. Davis and Simmt (2006) started by asking the question, "What is multiplication?" Many ideas and different meanings of multiplication emerged from the participants, including repeated addition; equal grouping; number-line hopping; sequential folding; many layered (literal meaning of 'multiply'); ratios and rates; array-generating; area-producing; dimension-changing; number-line stretching or compressing (Davis & Simmt, 2006, p. 301). Their studies with in-service teachers suggest that concept study is important in mathematics teaching and learning.

4. Connecting concept study with systems of complexity theory

For a deep understanding of mathematics concepts through concept study, it is important to know the layers/structures of concept study. There are five such layers, as we can see in the work of Davis and Renert (2014). These layers include: realizations, landscapes, entailments, conceptual blending and pedagogical problem solving. In this paper, the five layers of concept study are connected with complex systems of complexity theory, including openness, emergence, self-organization, organized randomness, diversity, interconnectedness, social interactions, and dynamic. They are described in the paragraphs below.

In concept study, the term realizations simply refers to meanings that can emerge from a certain concept. For instance, Davis and Simmt (2006) worked with teachers on multiplication in small groups through posing the question, "What is multiplication?" Teachers were able to generate extant meanings about multiplications within 30 minutes. Knowledge and knowing from teachers on the extant meanings they developed are shown in the figure below (Davis & Renert, 2014, 61).

Some Realizations of Multiplication

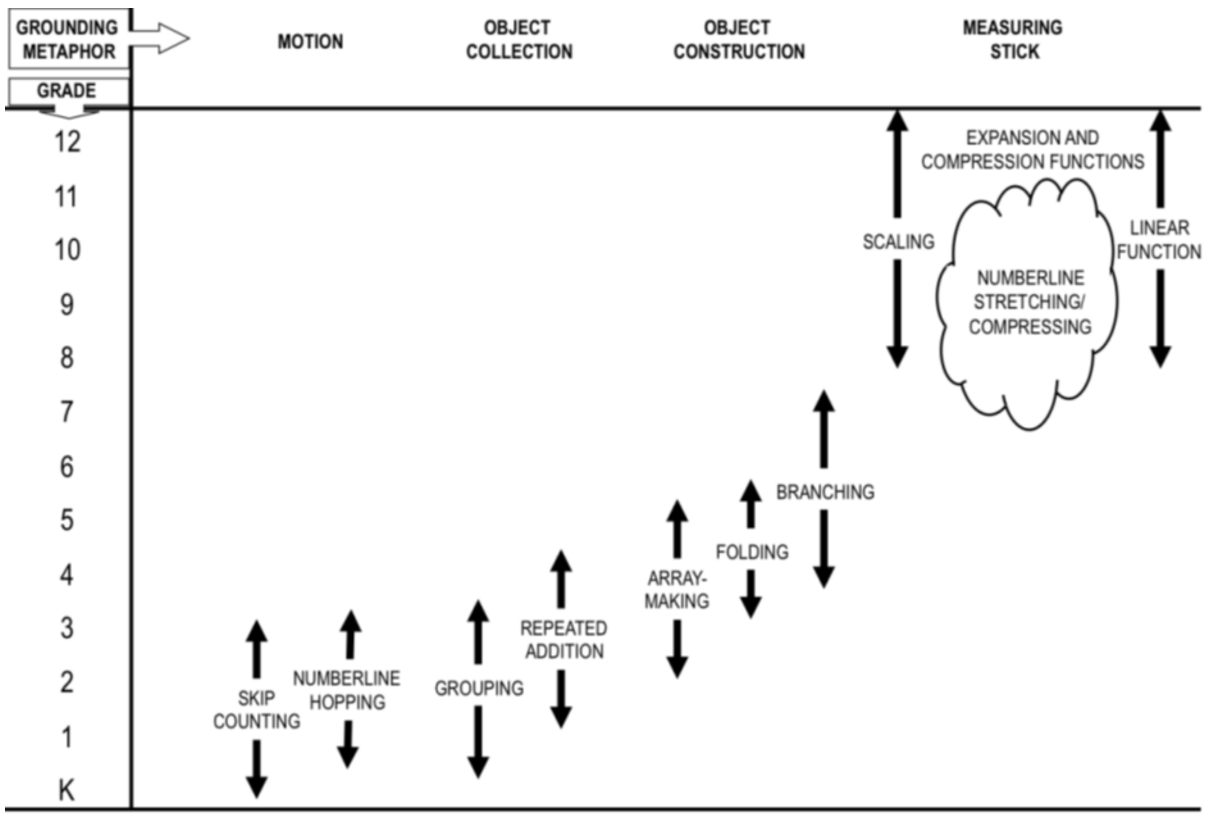
- grouping process
- repeated addition
- times-ing
- expanding (i.e., distributing across factors; e.g., how can you write $(x + 3)$, $(x + 2)$ times?)
- scaling
- repeated measures
- making areas (continuous)
- making arrays (discrete)
- proportional/steady increase/slope/rise
- splitting, folding, branching, sharing, and other 'dividings' (i.e., multiplyings)
- skip counting (jumping along a number line)
- transformations
- stretching/compressing a number line

Lingering worries:

- none of these on their own seem to do much to illuminate -2×-3
- what's going on when units are introduced – e.g., 3 g/L × 5 L?
- most of the entries rely on/suggest a linear or rectilinear basis/image of multiplication, which might work through middle school, but won't stretch across some things encountered in and beyond high school

The realized mathematical ideas above are a result of having a diversity of teachers having different learning abilities, understanding and teaching experiences. Also, mathematical ideas about multiplication emerged among teachers during small and large discussions. This is also associated by random organization of mathematical ideas about multiplication during small group and large discussions.

Landscapes involve an analysis of the flow of the extant meanings that are developed by teachers within a curriculum (Davis & Renert, 2014). For instance, from above realizations of multiplication, and referring to K-12 Curriculum, we find that the notions of grouping, skipping and numbering hopping occur in grades 1 through 3, notion of repeating addition occur in grades1 through 4, the notion of array making occurs in grades 2 through 5, and so on. The landscapes from the extant meanings developed by teachers are shown in the figure below (Davis & Renert, 2014, 64).



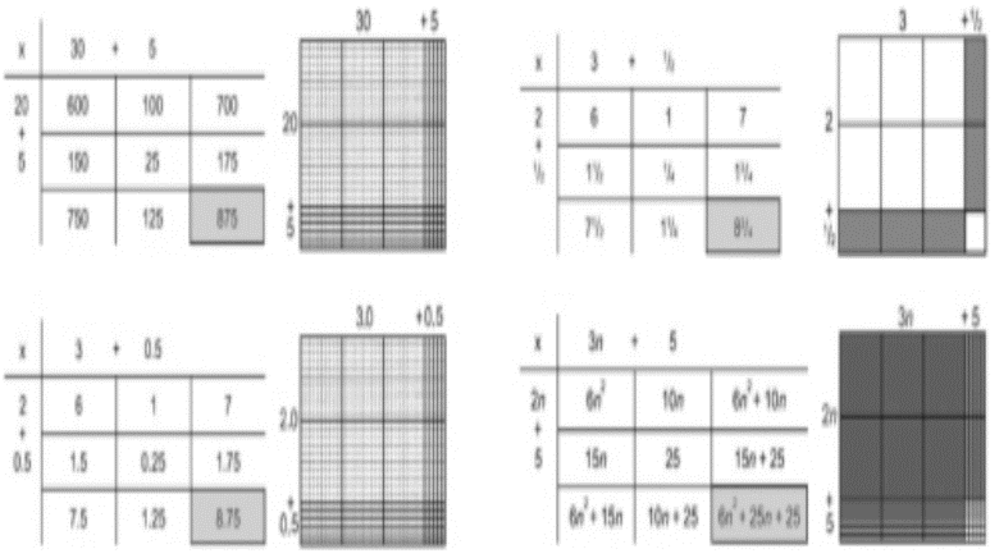
The above analysis of the realizations of multiplication is a result of interconnectedness of mathematical ideas in the curriculum. This interconnectedness is due to emergency/diversity of mathematical ideas among teachers during small group and large discussions about the meaning of multiplication.

Entailments refers to connecting the analogical realizations with other related concepts and topics. For example, from the realizations developed by Davis and Simmt (2006) working with teachers on the concept of multiplication, teachers and Davis noticed that, if multiplication is a repeated addition, then addend is a factor, a sum is a product, and the sum of ones is a prime. The more descriptions about the analogical implications of the extant meanings of multiplication are shown in the figure below (Davis & Renert, 2014, p. 69).

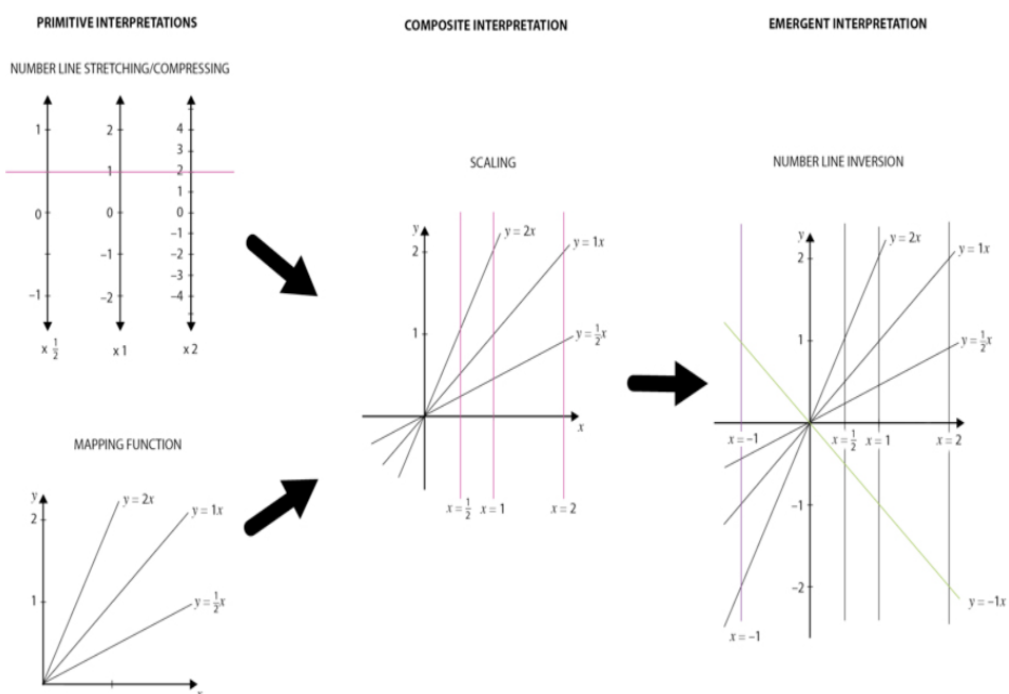
If multiplying is ...	... a factor is ...	... a product is ...	... commutativity is ...	... a prime is ... (necessary conditions, but not sufficient)
REPEATED ADDITION	ADDEND or NUMBER OF ADDENDS (2 × 3: 2 added to itself 3 times or vice versa)	a SUM	2 + 2 + 2 = 3 + 3	sum of ones
REPEATED GROUPING	NUMBER OF GROUPS or NUMBER OF ELEMENTS IN EACH GROUP	a SUM: total number of all the elements in the groups (cardinality of the set)	2 groups of 3 = 3 groups of 2	one group, or one element in each group
MAKING A GRID OR RECTANGULAR ARRAY	DIMENSION: number of rows (number in each column) and number of columns (number in each row)	NUMBER of cells	90°-DEGREE ROTATION (a 2-by-3 grid has the same number of cells as a 3-by-2)	one of the dimensions has to be 1
SKIP COUNTING	SIZE OF THE JUMP and NUMBER OF JUMPS	END DESTINATION (the last number you land on)	*a* jumps of distance *b* lands you in the same place as *b* jumps of length *a*	must make only a single jump or jump one space at a time
SCALING	SCALE FACTOR and ORIGINAL MEASURE	MEASURE OF THE FINAL MAGNIFICATION/ REDUCTION	size *a* scaled by a factor of *b* gives the same result as size *b* scaled by a factor of size *a*	when a magnification/ reduction can only be reached in unit increments or directly
AREA GENERATION	DIMENSIONS (lengths and widths)	AREA	90°-ROTATION: $lw = wl$	one dimension must be 1
NUMBER-LINE STRETCHING AND COMPRESSING	SCALE FACTOR and STARTING POSITION ON UNALTERED NUMBER LINE	CORRESPONDING POSITION ON STRETCHED/ COMPRESSED NUMBER LINE	If *c* corresponds to *point a* when line is scaled by *b*, it will correspond to *point b* when scalar is *a*	to get to *point c*, you must EITHER start at 1 with a scalar of c OR vice versa
FOLDING	NUMBER OF HORIZONTAL AND VERTICAL DIVISIONS (made by the folds)	NUMBER OF LAYERS	folding into *a* layers, then into *b* layers gives the same number of layers as *b* first, then *a*	can only be folded directly using $a - 1$ folds
BRANCHING	NUMBER OF STEMS and NUMBER OF BRANCHES PER STEM	TOTAL NUMBER OF BRANCHES AT THE LAST LEVEL	*a* branches of *b* stems has the same product as *b* branches of *a* stems	must either have 1 stem or 1 branch/stem
LINEAR FUNCTION $y = mx$	SLOPE and *x*-COORDINATE	*y*-COORDINATE	If $y = c$ when $m = a$ and $x = b$, then $y = c$ when $m = b$ and $x = a$.	To get to the desired *y*-coordinate *c*, either $m = c$ and $x = 1$ or $m = 1$ and $x = c$.

The above entailed mathematical ideas about multiplication are a result of dynamic understandings among teachers during small group and class discussions. Also, the ability of the individual teachers and group to see the interconnectedness of the generated ideas about the meaning of multiplication during small group and large discussions.

Conceptual blending involves blending the realizations into more powerful constructs (Davis & Renert, 2014). For instance, Davis and Simmt (2006) from a concept study about multiplication, used the combination of a grid-based algorithm and area-based image in highlighting some connections among standard algorithms for multiplying decimal fractions, multi-digit whole, and binomials as shown in the figure below (Davis & Renert, 2014, p. 73).



Another blended realization with meta-representation can be seen using graphs to blend linear models of multiplication as we can see in the figure below (Davis & Renert, 2014, p. 74).



The conceptual blending above is a result of openness of mathematical ideas about multiplication generated by teachers during small group and class discussions. Also, the resulted conceptual blending is due to a diversity of mathematical ideas emerged about the meaning of multiplication during small group and large discussions.

Pedagogical problem solving is the last layer of a concept study. Pedagogical problem solving involves concepts/ideas that can emerge when teachers meet together to encounter problems on a certain concept that occur when teaching to gain a deep understanding about the concept. Teachers are able to identify and help students with common errors and misconceptions. This strategy for identifying emergent mathematics helps teachers to develop a set of questions for the next concept study about a certain concept. For instance, Davis and Simmt (2006) after working with teachers about the concept of multiplication, teachers in the focused group were able to develop some questions for discussion in the next concept study session to gain a deep understanding of the concepts. The questions developed by teachers include: Is ∞ a number? What does it mean to divide by zero? And what is the difference between undefined and indefinite? From these questions developed by teachers, I can say that the knowledge and knowing of the pedagogical problem solving is "more complex than the study of solitary concepts" (Davis & Renert, 2014, p. 77). These emergent mathematics questions from teachers are due to teachers experiences through interacting with students in the classroom—"realistic settings" (Wheelahan, 2012, p. 4).

Pedagogical problem solving is a result of social interactions and diversity among teachers whereby individuals can realize what needs to be elaborated and how to foster collective understandings of a mathematical concept.

5. Knowledge and knowing of what to discuss through concept study under the lens of complexity theory

In mathematics, challenging concepts to be conducted as a concept study usually emerge from in-service teachers as they spend most of the time teaching students and interacting with students' mathematics learning, including marking students' assignment, tests and examinations. By teaching and marking tests, assignments and examinations, teachers get to know the concepts that are not clear to students as well as to teachers themselves. Knowing concept(s) that are difficult to the students, the identified difficulty concepts can be conducted as concept studies to allow teachers to gain a deep understanding of the concept through interacting with the community of expert teachers in mathematics.

Another way of getting a concept to be conducted for a deep understanding is through literature review of various mathematics texts including books. For instance, through literature review from Davis and Renert (2014), I got to know various concepts that have been studied as concept studies in the North American context, including what is multiplication?; is ∞ a number?; what does it mean to divide by zero?; what is the difference between undefined and indefinite?; what is interesting about circles?; why is the number π so mysterious?; are circles efficient?; how many sides does a circle have?; and what is function? I remember my first time to get to know about the concept study is when I was taking the EDSE 501 course in the last winter term during my first year doctoral studies in mathematics education. The EDSE 501 course was taught by Professor Elaine Simmt in our department of secondary education. I noticed the concept study to be a powerful model to gain a deep understanding of mathematics concepts. Not only getting knowledge on how to conduct a concept study from EDSE 501. I was able to conduct a concept study in the EDSE 501 about pi. Thus, the knowledge and knowing about the concept study came through taking this EDSE 501 course, participating in, and facilitating a concept study about pi in EDSE 501 class.

6. Advantages of conducting concept studies in mathematics education

There are various advantages that teachers and students can gain through concept studies, including expansive and creative thinking among teachers and students various mathematics concepts; a deep understanding of the concept among teachers and students; connecting the pre-knowledge with the existing/coming knowledge or connecting ideas from different mathematics topics; enhancing teachers' ability in designing mathematical tasks; enhancing teachers' knowledge of analysing students' common errors, conceptions and misconceptions; concept study can be used as an intervention; and enhancing teachers' ability in assessing students' work. These mentioned advantages can be described in more detail below.

6.1. Deep understanding of the concept

Teachers and students learn a concept in details. Conducting a concept study takes a lot of time discussing to a concept. I remember in March, 2015, when I was conducting the concept study about pi in EDSE 501 class. A class of graduate students and our instructor, we started with hands on activity in exploring the concept of pi. After that, I asked the participants, what is π? To start the conversation in a collective learning. The ongoing conversation generated open definitions of π as shown below.

Ratio of circumference and diameter of a circle

Irrational number which is infinite e.g. 3.14…

Approximated by a fraction 22/7.

Mathematical constant.

A symbol.

Ratio of measurements of the circumference and diameter of a circle.

A ratio between 3 1/7 and 3 10/71 (Archimedes)

Non-repeating decimal and non-terminated.

Radian measure of half a circle equivalent to 180°

Approximated by a decimal number 3.14.

From the above realizations of pi, as graduate students we gained a deep understanding of the meanings of π.

6.2. Expansive and creative thinking among teachers and students

Conducting a concept study in mathematics education usually involves a community of expert working together collaboratively and in a collective ways on a certain concept. As the result, some of the participants become more creative in thinking about a concept and expanding ideas on the concept through ongoing conversations—emergent mathematics.

6.3. Connecting the pre-knowledge with the existing or the coming knowledge

Concept study can help teachers to connect their previous knowledge with the existing/coming knowledge. For instance, as a result of working together on the concept of pi in the EDSE 501 class, participants were able to demonstrate detailed knowledge of the K-12 experience and the connectedness of the concept of pi in upper and lower grades. In a discussion, participants pointed out that students need to be familiar with different concepts before learning a concept of pi, including place value-repeating and non-repeating; the shape of a circle; measurements-to get the skills and accuracy of measuring the diameter and circumference of a circle; meaning of diameter and circumference; division of numbers-whole number and decimal numbers; a constant and variable; and irrational numbers and rational numbers. Furthermore, through the discussion about pi, the participants noticed that in upper grades, students expect to know and use pi in different topics, including complex number; trigonometry functions—for instance in plotting the graphs; solving the trigonometric equations for exact value; volume of 3-D objects such as sphere, cylinder and cone; radian measure; and surface areas of 3-D objects such as sphere, cylinder and sphere.

6.4. Concept study as an intervention

Concept study can be used as an intervention in mathematics teaching and learning contexts, such as to the classroom that a teacher uses traditional teaching approach including lecturing. Concept study offers collective knowledge among participants—

teachers and students. Concept study as an approach eliminates the gap between knowers and learners. Using concept study, learning becomes learner centered in a collective learning. Thus, concept study can replace the traditional method of teaching whereby teachers are the only sources of knowledge, and students are treated as receivers of knowledge. Therefore, instead of using lecturing in teaching mathematics class, teacher can use a concept study to give power, democracy and voice among students in mathematics learning context. In Tanzania, lecturing is mostly used in teaching mathematics. I think there is a need to shift by allowing collective learning to take place in the class through using learner centered approaches, including concept study. This model gives an opportunity for other countries including Tanzania to engage teachers in improving their mathematics teaching and learning in schools, teacher colleges, and universities.

6.5. Enhancing teachers' ability in designing mathematical tasks

Designing a task involves teachers' ability focusing on the kinds of students' learning outcomes needed to be measured. In designing a task, teachers need to ask themselves the questions, "in what extent and how?" The task framework for the teacher focuses on teacher learning as the destination. To maintain the task on specialized content knowledge, SCK, the designed task should focus on explanation, representations and building correspondences among them, core mathematical ideas, pulling the mathematics out of teachers' comments, engaging in teaching practices that use SCK, and making explicit connections to the work of teaching. The designed task should have the following features: unpacking, making explicit, and developing a flexible understanding of mathematical ideas that are central to the school curriculum; opening opportunities to build connections among mathematical ideas; provoking a stumble due to a superficial "understanding" of an idea; lending itself to alternative/multiple representations and solution methods; and providing opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis, posing questions, writing on the board). The question below is among well designed tasks for SCK goal.

I bought 10 oranges and 8 mangoes for $3.32. One mango and one orange cost $.80. How much does one mango and one orange each cost?

Work individually and when ready share with your group (teachers):

What are the different ways you thought about the task? What are the different ways the solution was represented? Which are similar? Which are different? What do you think to be the best approach in solving this problem? Why?

6.6. Enhancing teachers' knowledge of analyzing students' common errors, conceptions, and misconceptions

Knowledge of analyzing students' common errors, conceptions and misconceptions requires a teacher to have SCK. The teacher is able to identify students' errors by diagnosing through giving them tasks to do when teaching. Asking students questions in the class helps the teacher to realize whether students understand the concepts or they have misconceptions. Also, a teacher's ability to analyze different approaches coming out from the students when solving a given mathematical task. Having mathematical knowledge, brings me to think on the question pointed out by Zembat (2012) when conducting the research to see the extent on how teachers in public schools in United Arab Emirates (UAE) are able to analyze the students' work. Zembat presented the following question:

A computer game store is having a sale. They have advertised 10% off everything in the store. They also have just purchased a new shipment of computer games. These games cost the store 32.11AED each. They want to price the game so that they will make at least a 40% profit, even at the sale price. What is the lowest regular selling price for the game that will allow this profit? (Bair & Rich, 2011)

a. If a student brings this question to your mathematics class, to what extend would you feel confident (or comfortable) in analyzing this problem situation before you actually solve it?

b. What mathematical knowledge or understandings are required to solve this problem? [Please be specific in your answer. For example, saying that it requires algebra or geometry is not informative! If you need, you can solve the problem here and then talk about the mathematical components of it!]

c. What common mathematical errors (or mistakes) would you expect from students in solving such a problem?

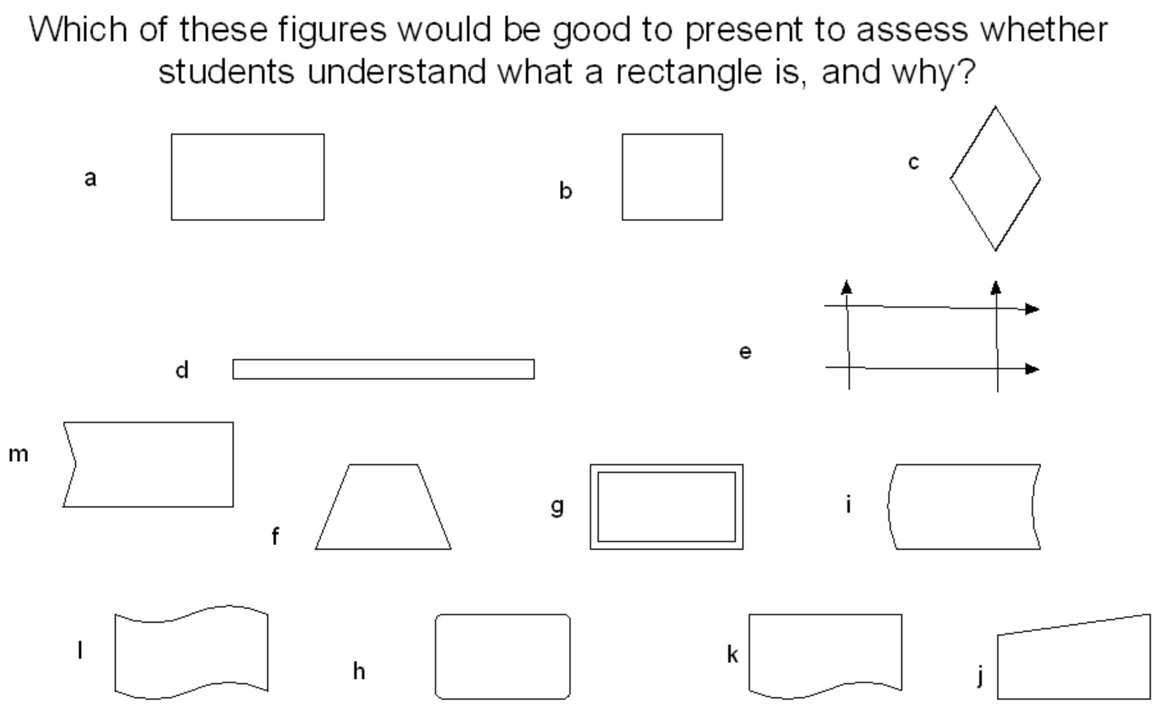
(Zembat,2012, para.16)

Zembat (2012) found that most of the participants solved the problem first, and then talked superficially about the possible student errors in calculation. I think, this question does not require solving the problem first but thinking and focusing on the student errors that are likely to happen. The students' errors can be easily recognized by the teacher through listening to the student ideas, interpreting those ideas, and thinking about what it means to think like a student in certain problem situations. Listening to students is not a simple task to the teacher as it involves evaluative, interpretive and hermeneutics listening. Therefore, listening is embodied in the nested circle as we can see in the work of Devis and Renert (2014, p. 86). Thus, as teachers, we are required to listen to students by articulating all these three complicated mode of listening.

6.7. Enhancing teachers' ability in assessing students' work

Assessing students' work is the activity of the teacher to design and implement the assessment activities to be able to know individual student's mathematical ability on a certain concept(s). Thus, mathematics teachers' understanding of assessment involve teachers being able to create/choose better the assessment items for testing student understanding of a certain mathematical concept. The assessment should reflect individual student's ability in terms of understanding the concepts, common errors, and misconceptions. Hence, as teachers, the assessment helps us to figure out how and to what extent students understand the concept. Doing it this way, teachers get the sense to know the strength and weakness of the individual students to the targeted concepts, and learning the way to help those students who might end up with misconceptions through tracing individual student's errors.

I figured out how teacher's ability to assess the task first is important before assigning the task to students, through creating an example below.



Teachers should have ability to know and articulate the ideas in terms of questions to be able to assess well the students in any mathematical context. This kind of knowledge for the teacher can be seen through reflecting on the question below created by Ball and her colleagues (2010) in their work tilted, "Designing and Using Mathematical Tasks to Develop Specialized Content Knowledge for Teaching"

Which of these is a mathematically accurate definition of "rectangle"?

A rectangle is a figure with four straight sides, two long and two shorter.

A rectangle is a shape with exactly four connected straight line segments.

A rectangle is flat, and has four straight line segments, four square corners, and it is closed all the way around.

For any that are not mathematically accurate, give an example that shows what is wrong. *(Ball, et al., 2010, p. 7)*

From the question above, it helps the teacher to be able to articulate the ideas of a triangle and assessing each definition through thinking and reasoning in multiple ways.

Assessing teachers' SCK can be seen in the work of Ball, et al. (2005), whereby Ball and her colleagues designed and presented the multiplication problem for the teachers to assess three different approaches used in solving the problem 35 ×25 for grade three children. The item question designed by Ball and her colleagues (2005) is shown below.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	*Student B*	*Student C*
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+700	150
875	875	100
		+600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would not work for all whole numbers	I' m not sure
a) Method A	1	2	3
b Method B	1	2	3
c Method C	1	2	3

(Ball, et al., 2005, p. 402)

The item above does not direct what the teacher should teach in the class or what students should be able to know, but, it asks a mathematical question about alternate solution methods that teachers could know, to describe an important skill and knowledge for effective teaching of the concept of multiplication of numbers.

Concept study under the lens of complexity theory has proved to be productive to the teachers and students for a deep understanding of the concepts and professional development to in-service and novice teachers (Davis & Simmt, 2006; Davis & Renert, 2014, Davis, 2008). However, when conducting concept studies there are some challenges that can emerge, including demanding: more time for deep understanding of the concept; various learning resources and facilities such as internet services, round table classroom sitting arrangements; commitment among participants; collective learning that can allow emergent of ideas; and self-disciplines and collaboration among participants to avoid rupture of the dialogues/discussions about the concept—think, pair and share.

7. Conclusions

There are different forms of knowledge, and different ways of coming to know the truth. In this paper, the notions of knowledge and knowing in mathematics through concept study in relation to systems of complexity theory, have implications to the teachers, students, curriculum developers and other educational stake holders in different parts of the world for conceptual development and understanding and organization of the class. I believe that using concept study under the lens of complexity theory serves the purpose of mathematics education to the students in the sense that "the purpose of education is to help equip students with the knowledge and capabilities they need to make their way in the world" (Wheelahan, 2012, p. 70) and the truth being a normative goal of curriculum (Wheelahan, 2012). Thus students' voice, power, social conversation and democracy shape the learning outcomes as well as the access to the learning and the type of learning that takes place in the realistic settings—in this case, concept study under the lens of complexity theory in the mathematics classroom.

References

Ball, D., Sleep, L., Suzuka, K., Bass, H., Thames, M., & Lewis, J. (2010). Designing and using mathematical tasks to develop specialized content knowledge for teaching. RMLL Researching Mathematics Leader Learning, 1-39. **http://www.bing.com/search?q=Designing+and+Using+Mathematical+Tasks+to+Develop+Specialized+Content+Knowledge+for+Teaching+&qs=n&form=QBRE&pq=designing+and+using+mathematical+tasks+to+develop+specialized+content+knowledge+for+teaching+&sc=0-0&sp=-1&sk=&cvid=8f9f126587d34750aa40536bf382520a**

Ball, D. L., Hill, H. C., & Rowan, B. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42 (2), 371-406.

Davis, B. (2008) Is 1 a prime number? Developing teacher knowledge through concept study.

Mathematics Teaching in the Middle School (NCTM), 14(2), 86–91.

Davis, B. (2004). Inventions of teaching: A genealogy. New York, NY: Routledge.

Davis, B., & Renert, M. (2014). The math teachers know: Profound understanding of emergent mathematics. New York, NY: Routledge.

Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: an ongoing investigation of the mathematics that teachers (need to) know. Educational Studies in Mathematics, 61, 293-319. **http://www.jstor.org/discover/10.2307/25472073?sid=21105242935431&uid=3737720&uid=4&uid=2&uid=3739392**

Davis, B., & Sumara, D. (2006). Complexity and education: Inquiries into learning, teaching, and research. Mahwah, NJ: Lawrence Erlbaum.

Doll, W. E. (1993). A post-modern perspective on curriculum. New York: Teachers College Press.

Fernandez, C., & Yoshida, M. (2004). Lesson study: a Japanese approach to improving mathematics teaching and learning. Mahwah, NJ: Erlbaum.

Newell, C. (2008). The class as a learning entity (complex adaptive system): An idea from complexity science and educational research. Simon Fraser University Educational Review 2(1), 5-17.

Usiskin, Z., Peressini, A., Marchisotto, E.A., & Stanley, D. (2003). Mathematics for high school teachers: An advanced perspective. Upper Saddle River, NJ: Pearson.

Wheelahan, L. (2012). Why knowledge matters in curriculum: A social realist argument. New York, NY: Routledge.

Zembat, O. I. (2012). Specialized content knowledge of mathematics teachers in UAE context. Mevlana (Rumi) University, Turkey, 1-10. **http://www.bing.com/search?q=SPECIALIZED+CONTENT+KNOWLEDGE+OF+MATHEMATICS+TEACHERS+IN+UAE+CONTEXT+&qs=n&form=QBRE&pq=specialized+content+knowledge+of+mathematics+teachers+in+uae+context+&sc=0-28&sp=-1&sk=&cvid=95ebfb23c867484298ecd37779c1991b**
