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How pre-service teachers perceive geometric reflection in a dynamic environment: Motion view and mapping view

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Abstract

Starting from the elementary school years, geometric reflection has potential for students to improve their understanding of plane geometry. Although it gives them opportunities to investigate the motion view of the transformed figures and mapping of all points in the plane onto itself, the instructional tasks and teaching practices generally focus on developing motion views and ignore the mapping view. The purpose of this study is to examine how PSTs understand the geometric reflection in terms of a motion view and mapping view by using a dynamic geometry software (DGS) (GeoGebra), and in what ways the GeoGebra support or limit pre-service teachers' understanding of geometric reflection in terms of motion and mapping views. Four case studies were constructed from transcript audio records, and videos. The data were analyzed using motion and mapping views as inferred from related studies. The results indicated that PSTs' views were limited to a motion view, and they could not show improvement in their understanding into a mapping view due to the use of GeoGebra and its tools (e.g., reflect about line, segment, dragging).

Keywords: Geometric reflection; motion view; mapping view; dynamic geometry software

1. Introduction

The use of technology in the teaching and learning of geometry has gained importance in the last two decades. Among such technological tools, dynamic geometry software (DGS) is commonly used, such as Geometers' Sketchpad, GeoGebra, and Cabri (Falcade et al., 2007; Gawlick, 2002; Hollebrands, 2003; 2007; Hölzl, 1996; Laborde, 2001; 2005; Ruthven et al., 2008; Tikoo, 1998; Yanik, 2013; Yanik & Flores, 2009). A synthesis of literature on the role of DGS in mathematics education reveals that these tools support

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student understanding (Hohenwarter et al., 2009; Hollebrands, 2003; 2007; Sinclair & Crespo, 2006; Yanik, 2013; Yanik & Flores, 2009), including geometric reflection (Hollebrands, 2003, 2007; Yanik, 2011, 2013).

Studying geometric reflection provides a foundation for students to understand other mathematical concepts including functions, symmetry, and congruence (Hollebrands, 2003; Yanik, 2006); to develop visualization and analytic strategies (Boulter & Kirby, 1994); to develop cognitive skills, such as critical thinking, problem-solving, conjecturing, and deductive reasoning (Jones, 2002); and to develop mathematical skills, such as describing patterns, discovering basic features of isometries, making generalizations, and developing spatial competencies (Clements et al. 1997; Portnoy et al., 2006). Therefore, the teaching and learning of geometric reflection in geometry can have a strong positive impact on students' understanding of mathematics in general.

Understanding geometric reflection has been conceptualized in terms of two broad notions, "motion view" and "mapping view" (Akarsu, 2022; Edwards, 1997; Hollebrands, 2003; Mhlolo & Schafer, 2013; Yanik, 2011, Yanik & Flores, 2009). A person with a motion view has challenges understanding the role of the reflection line (e.g., using equidistance and perpendicular properties to determine relations between pre-image and image figures and the reflection line), considers the domain as a single geometric figure, and sees the plane as a background that is separate from the focal figure, which alone can be manipulated. This view represents reflection inaccurately or at best incompletely. Hollebrands (2004) described an alternative understanding of geometric reflection, informed by formal mathematics, as "mapping," in which "A transformation on the plane is a one-to-one correspondence from the set of points in the plane onto itself" (Martin, 1982, p. 1). From this viewpoint, a person with a mapping view knows the role of the reflection line and sees the domain as all points in the plane, of which the geometric figure is part.

The transition from a motion view to a mapping view is important for understanding the concept of geometric reflection (Akarsu, 2022; Hollebrands, 2003, 2007; Mhloo & Schafer, 2013). Previous researchers have suggested that learners usually understand geometric reflection from a motion view, which obstructs the mapping view (Hollebrands, 2003, 2007; Mhloo & Schafer, 2013; Yanik, 2006). In particular, recent studies have identified the role of the reflection line, domain, and plane as important sub-concepts to understand in order to progress from a motion view to a mapping view of geometric reflection (Yanik & Flores, 2009). Further exploration of strategies learners use in developing understanding of geometric reflection is needed to help them move beyond motion view. For this purpose, we assumed that DGS could be an effective tool for learners to move from a motion view to a mapping view. Based on this premise, this study was an investigation of how PSTs understand geometric reflection in terms of motion view and mapping views when using GeoGebra, a dynamic geometry software

(DGS), and in what ways GeoGebra supported or limited PSTs' understanding of geometric reflection in terms of both views.

2. Literature Review

2.1. Ways in which Geogebra mediates learners' understanding of geometric reflection

DGS provides opportunities for students to create objects and act upon them, such as drawing, constructing, and measuring (Hollebrands, 2007; Zbiek et al., 2007); and recognizing patterns, making conjectures, and formulating conclusions (Tikoo, 1998; Yanik, 2013). Hollebrands (2007) found that when students interact with DGS, they observe and experience geometric reflection, providing opportunities for them to learn by making connections between representations (Hahkioniemi, 2013; Clements et al., 2008; Zbiek et al., 2007).

According to van Voorst (1999), DGS can be "useful in helping students view mathematics less passively as a set of procedures, and more actively as reasoning, exploring, solving problems, generating new information, and asking new questions" (p. 2). Therefore, to promote learners' reasoning, it is important to know how they interact with DGS programs and how these interactions influence their understanding of geometrical reflection. Learners are expected to focus on the role of the reflection line, the behaviors of the pre-image, and image points in the reflection when they use features of GeoGebra, such as the dragging modality and measurement capabilities, rather than reasoning only about the appearance of the physical representations.

To sum up, use of GeoGebra provides opportunities for learners to strengthen their understanding of geometric reflection. In particular, the dragging and measurement features of GeoGebra are helpful for students to explore the properties and definition of a reflection, understand the role of the reflection line, make and test conjectures, and construct new understanding. Also, dragging might help students to determine what properties remain invariant under reflection. With the perspectives gathered from the findings of the studies mentioned above, PSTs' understanding of the concept of reflection was investigated based on their actions with DGS.

2.2. Students' and pre-service teachers' understanding of geometric reflection

Previous researchers have found that students and PSTs tend to have a motion view of geometric reflection as indicated by their difficulties with the role of reflection line, conception of the domain as a single figure, and perception of geometric figures as separate from the plane, not a subset of the plane (Edwards, 2003; Edwards & Zaskis, 1993; Hollebrands, 2003, 2004; Portnoy et al., 2006; Yanik & Flores, 2009; Yanik, 2011). Edwards (2003) found that middle school, high school, and college students all had a motion view of reflection as they considered geometric figures as moveable on the plane rather than as part of the plane. That is, when they performed a geometric reflection, they thought of it as moving some points or a figure to another place, which is evidence that students have a motion view of reflection.

Flanagan (2001) worked with high school students to investigate their understanding of translation, rotation, reflection, and dilation using the Geometer's Sketchpad (GSP). She found that students consider the domain as single points or figures, which are separate from the plane. For instance, Flanagan asked a student to create points A, B, C, and D using GSP and apply a reflection over the y-axis (i.e., image points, A', B', C', D'). Then, she asked him to predict what would change if the reflection line were dragged. He stated that if the reflection line were dragged, the pre-image and image points would move the same distance and direction as the reflection line. To test this conjecture, he dragged the reflection line and was surprised to find that the pre-image points did not move; only the image points moved. This response exemplified that, first, students considered that geometric reflection reflects the given points or the figure rather than all points in the plane, and second, students observed points as moveable on the plane rather than as a part of the plane. Therefore, use of the dragging features of GSP supported students' motion view of domain and plane rather than a mapping view of domain and plane.

Yanik (2006) investigated four PSTs' understanding of geometric transformations (e.g., translation, reflection, rotation, and dilation) using GSP and found they had a motion view of geometric reflection. In particular, all participants considered the domain as a single figure and had difficulties describing the geometric reflection and applying the definition of the plane in geometric reflection. One challenge for the PSTs was that the use of GSP may support PSTs' understanding of the domain as a single figure as they need to select a figure from the domain to apply a reflection. Using the technology to select a figure from the domain may support PSTs' tendency to think of geometric reflection as applied to that particular figure, which reinforces the motion view.

To sum up, previous studies show that students and PSTs tend to have a motion perspective in their conceptualizations of the sub-concepts of geometric reflection such as reflection line, domain, and plane. It is also found that there are disadvantages as well as advantages in the use of dynamic geometry software for understanding these sub-concepts. The purpose of the study was to extend the literature by investigating the PSTs' perspectives on geometric reflection and to investigate how the use of GeoGebra affects these perspectives.

3. Method

3.1. Participants

Four pre-service teachers (PSTs) of mathematics, two males and two females, voluntarily participated in this study at a public university located in the eastern part of Turkey. They were in their fourth (last) year of a bachelor's degree program in the department of middle school mathematics education. Three criteria were used in the selection of the participants: First, their willingness to participate; second, their possession of at least a basic level of knowledge about geometric reflection and using dynamic geometry environments at least at a basic level; and third their ability to articulate their thought processes. All had successfully completed courses in calculus, analytic geometry, teaching geometry in which they covered geometric reflections over a specific point (or origin), x- and y- axes, and line x=y. In addition, all had taken an elective course, "Technology Supported Mathematics Education," in which they were taught how to use GeoGebra for preparing instructional materials, presenting mathematical activities, and doing basic geometric proofs. Therefore, they were fluent in using GeoGebra and familiar with its properties. In the reporting, the codes PST1 (female), PST2 (male), PST3 (male) and PST4 (female) were used to identify the participants and R was used for the researcher.

3.2. Data collection tools and process

This study was a follow-up another study (Akarsu, 2018) of four pre-service elementary mathematics teachers' understanding of geometric reflection based on the motion and mapping perspectives in a paper-pencil environment. The data were collected through semi-structured clinical interviews, which were based on the tasks used in Akarsu's (2018) study. One initial and three exploratory one-on-one interviews were conducted, lasting from 25 min to 41 min with an average of 34 minutes. The purpose of the initial interview was to gather participants' demographic and background information and elicit their initial ideas about geometric reflection. On the other hand, the first exploratory interview was semi-structured task-based clinical interview using GeoGebra to further examine each of the participants his/her initial understanding of geometric reflection and explore his/her reasoning and justifications for the tasks to identify motion and mapping views of reflection line, domain and plane (see Figure 1).

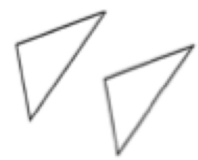
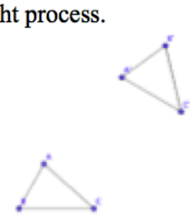
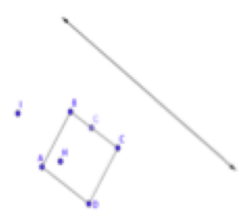
Interview	**Questions and Tasks**
First interview section	1.Did you learn geometric reflection 2.What did you learn about geometric reflection? 3.How you define the geometric reflection? 4.What are the properties of the geometric reflection? 5. Is this a reflection?  6. Can you find a reflection line? Explain your thought process.  7. Can you perform the reflection? How did you perform the reflection? What properties did you use for performing geometric reflection? 

Figure 1. Sample interview tasks for the first interview session

The second and third exploratory interviews focused mainly on the possibility of PSTs' transitioning from a motion to a mapping view. The PSTs were provided with GeoGebra files representing combinations of points, open or closed figures with or without highlighted inner or outer regions, and a reflection line positioned vertically, horizontally, or obliquely. The PSTs were then asked to find the reflections for the combination of points and figures with respect to the given reflection line (see Figure 2). Participants were also asked to respond to "how" and "why" questions in order to uncover their understanding about geometric reflection, specifically to determine whether PSTs' understanding reflected a motion view or mapping view in the interviews.

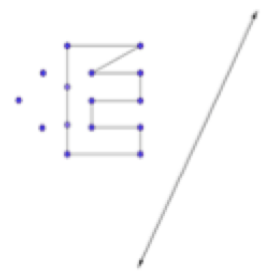
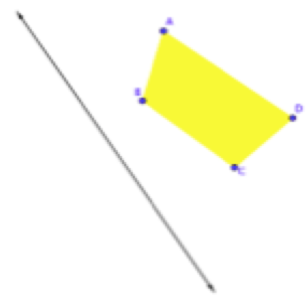
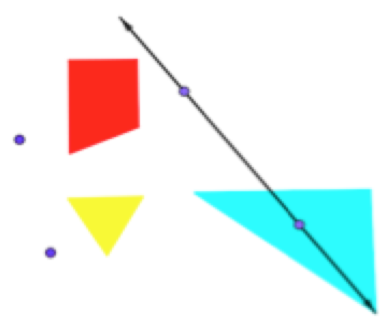
Interview	**Questions and Tasks**
Second interview section	1. Can you perform the reflection? How did you perform the reflection? What did you reflect? What properties did you use for performing geometric reflection?  2.  3. 

Figure 2. Sample interview tasks for the second third exploratory interviews

Each PST was interviewed twice on different days (covering initial and first exploratory interviews on the first day and second and third exploratory interviews on the other day) in the office of one of the researchers with both researchers present at all interview sessions. All GeoGebra files were uploaded to the researcher's computer, and at each interview session, the participant read the questions, answered them on this computer, and shared his/her reasoning aloud. Screen recorder software had been installed on this computer, so all the work the PSTs performed on the computer screen was recorded. In addition, all interviews were audio- and video-recorded and later transcribed for data analysis.

3.3. Data analysis

The data gathered from the interviews were analyzed concurrently with ongoing data collection. The main purpose in this analysis was to seek patterns and gather richer and detailed information about the development of the PSTs' thought processes regarding geometric reflection. First, all interviews were transcribed as Word documents, and both researchers separately viewed the video-recordings. The PSTs' baseline understandings of geometric reflection were determined in the initial interviews. Then, any developments in their understandings were documented in the evidence gathered from PSTs' responses using GeoGebra to questions posed in the task-based interviews. Whether PSTs exhibited motion or mapping views of geometric reflection were analyzed according to criteria proposed by Flanagan (2001), Yanik (2006), and Akarsu (2022), as summarized in Table 1. The researchers based their evaluations of PSTs' understanding of geometric reflection on evidence from their definitions and explanations.

Table 1. Indications of motion and mapping views for the concepts of reflection line, domain and plane in the geometric reflection

Concepts of geometric reflection	Motion view	Mapping view	Examples responses and actions to have mapping view
Reflection line (When performing a geometric reflection, ...)	PST does not use equidistance property to determine the place of the image figure	PST use the equidistance property to determine the place of the image figure	"The reflected point on triangle and its reflection should have equal distances to the reflection line"
	PST does not use perpendicularity property to determine the place of the image figure	PST use the perpendicularity property to determine the place of the image figure	"If we draw a line between these points (pre-image and image points) [on the rectangle figure], it should be perpendicular to this (reflection line)"
	PST reflect the figure as a whole rather than as a collection of points	PST reflect the figure as a collection of points	"I reflected the corner points of the triangle and connected them. Actually, I reflect all points on the perimeter of the triangle"
Domain *(When performing a reflection, …)*	PST cannot define plane (e.g., the plane is empty)	PST define plane	"There are infinitely many points on the plane"
	PST only considers given points or figures for performing geometric reflection	PST reflects all points in the plane as a domain rather than given points or figures	"I reflected all points left half of the plane to the right half of the plane, and right half of the plane to the left half of the plane"
Plane *(When performing a reflection, …)*	PST cannot define the plane	PST define plane	"There are infinitely many points on the plane"
	PSTs cannot use the definition of plane in the geometric reflection (the plane is empty and there is a movement of the points or figures from left half of the plane to the right half of the plane) by either considering the points forming the shape independent of the plane or the plane is empty	PST justifies that the plane is composed of infinitely many points AND there is no movement on the plane. The points or figures are subsets of the plane.	No example

As shown in Table 1, PSTs were considered to have a mapping view for each component of geometric reflection (i.e., reflection line, domain, and plane), if they demonstrated that understanding in their work or provided accurate justifications. With regard to the reflection line, for example, the PSTs had to either use the equidistance and perpendicularity properties of the geometric reflection and consider the figure as a collection of points for performing geometric reflection or provide justifications similar to those in Table 1, indicating their understanding that these properties are necessary to have a mapping view of the reflection line. In addition, they had to show the similarities of the figures before and after reflections on GeoGebra or justify their answers accordingly. If PSTs ignored or missed any of these criteria for a particular concept, the researchers decided that they had a motion view for it. Similar approaches were followed by the researchers for the domain and the plane concepts of the geometric reflection. The findings were organized concept by concept. In addition, interview excerpts were provided to support participants' understanding and development for the concepts of geometric reflection.

Both researchers separately analyzed transcripts of interviews with PSTs and viewed their works on GeoGebra environments. They looked for the instances that support PSTs' understanding of geometric reflection for these specific concepts with respect to the criteria in Table 1. The researchers created codes for each concept of geometric reflection, pointing out the PSTs' motion and/or mapping views. After the comparisons of the researchers' evaluations and discussions on the observed instances and created codes (whether they support PSTs' motion and/or mapping views), they determined PSTs' initial ways of thinking about and their final understandings of the reflection line, the domain, and the plane in the DGS environment. All findings were combined after the researchers reached full consensus on their evaluations.

4. Results

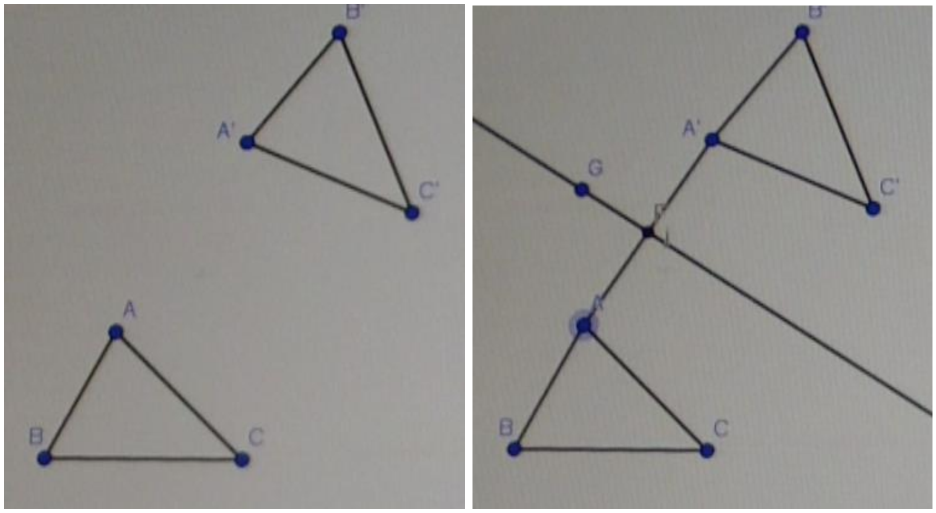
The purpose of this study was to investigate how four PSTs understood geometric reflection in terms of motion and mapping views and in what ways using GeoGebra supported or limited their understanding of geometric reflection. In this section, the findings regarding PSTs' initial understanding of the concepts of geometric reflection and the further development of their understanding are discussed separately.

4.1. PSTs' Understanding of Reflection Line

In the initial interview, when they were asked to define geometric reflection, PSTs' responses indicated a motion-view understanding of the reflection line. While PST1 emphasized the equidistance properties of geometric reflection, none of them specifically highlighted the perpendicularity properties in their definitions. Also, based on their definitions, it is not clear how they considered the relationship between the pre-image and image figures and the reflection line. In the initial interviews, the PSTs gave the following definitions of geometric reflection:

> PST1: It is kind of reflecting a point or a figure about an object [referring to the reflection line], I mean like a mirror… First, I reflected a line, then took a point and then reflected it about this line [the line constructed]. Its location has changed, and our point has crossed the line at the same rate [distance].
>
> PST2: In the geometric reflection, the magnitude and the form of a figure remains constant but the direction changes. I mean according to x-axis, y-axis. I know that the direction changes in the geometric reflection... But the figure is the same.
>
> PST3: When we say [geometric] reflection, something like seeing ourselves in the mirror… For example, if we reflect a point [on the figure to be reflected] about x-axis, the ordinate of a point [on the figure] becomes minus [changes sign]
>
> PST4: I think that it is the image reflected by a figure in the mirror, it is the figure reflected, that is, when an image is reflected in the coordinate plane and folded, it forms in the same way, overlapping each other.

In the first exploratory interview, they were given two figures without the reflection line and asked to find the reflection line (see Figure 3a). All PSTs used the segment tool to connect A to A`, then used a midpoint tool to find the center of the AA', and finally used a line tool to draw a line between A and A` (see Figure 3b).



(3a) (3b)

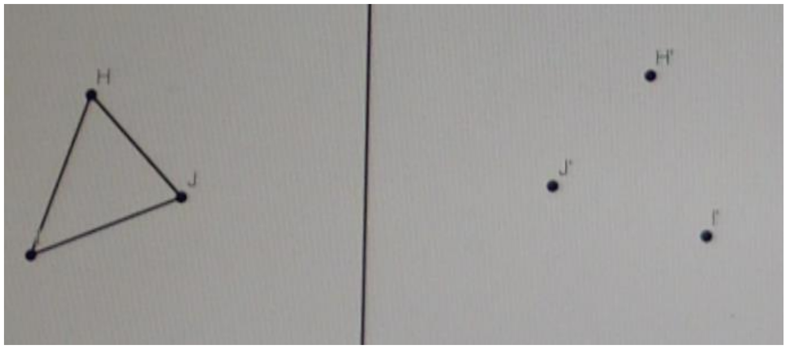
Figures 3a and 3b. PSTs' example tasks.

When asked what properties are important to perform a geometric reflection, the PSTs responded as follows:

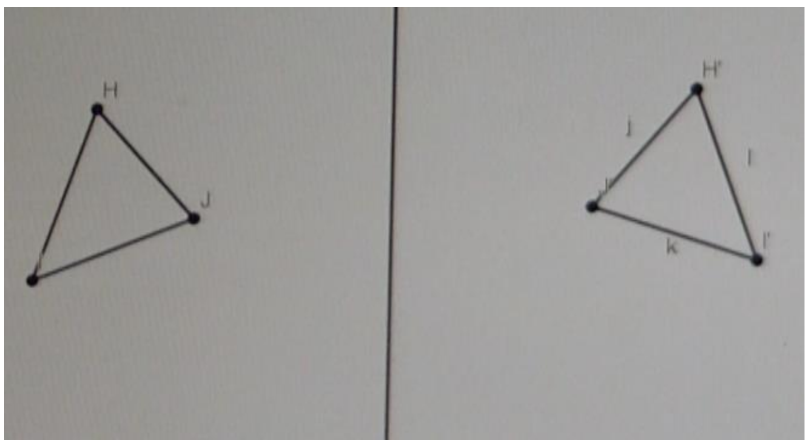
> PST1: First, we need a reflection line. Then, the distance between figures is important [referring to equidistance property between pre-image and image figure].
>
> PST2: Critical points [referring to corner points of the figure] are important. We can use the reflect-about-line tool to reflect the points.
>
> PST3: I used the reflect-about-line tool to reflect points.
>
> PST4: In Geogebra, I need to determine the corner points of the triangle. Then, I need to reflect three corner points by using the-reflect-about line tool. Finally, I used the segment tool to connect all three corners to get a triangle.

In their explanations, only PST1 mentioned the properties of equidistance and perpendicularity for finding a reflection line, while PST2, PST3, PST4 treated the figure as a collection of the points. We interpreted PST1's action as evidence that she considered the figure as a whole figure. Based on their explanations, we inferred that all PSTs had a motion view of the reflection line as they had difficulties using equidistance and perpendicularity properties to determine the relationship between the pre-image and image figures and the reflection line and in three cases did not mention these properties.

Later in the first exploratory interview, the PSTs were also asked to reflect a triangle over the oblique reflection in GeoGebra (see Figure 4a). All PSTs selected three corner points and used the reflect-about-line tool to reflect the three points with no evidence of understanding the meanings of properties of geometric reflection as they were focused on a motion view. They then used the segment tool to connect the three corner points to make a triangle (see Figure 4b).



(4a)



(4b)

Figures 4a and 4b. PSTs' drawing of a reflection on an oblique line

After reflecting the triangle, they were asked how to perform the geometric reflection and what properties are important for doing so, to which they responded as follows:

> PST1: I have taken the corner points and used the reflect-about-line tool to reflect the other side of the reflection line. The distance of the pre-image and image figure to the reflection line is important [referring to equidistance property].
>
> PST2: I reflected corner points by using the-reflect-about line tool and connected them. While performing a geometric reflection, the given figure and its location is important. Then, we can use the GeoGebra property to reflect it.
>
> PST3: I used the-reflect-about line tool to reflect points and segments of the triangle. I used the GeoGebra property.
>
> PST4: I reflected the corner points and segments of the triangle. Use of the reflect-about-line tool is important for performing geometric reflection.

They performed the reflection using GeoGebra. Except for PST1, they appeared to be unaware of the properties of the reflection that were preserved. Because they used the GeoGebra tools to perform the reflection without engaging in reasoning, they attended only to the physical representation of the figure on the screen. PSTs operating at this level of understanding exhibited uncertainty about how to use the properties of geometric reflection for performing a reflection. This observation highlights that reasoning about a geometric reflection depends on how it is defined and what properties are noticed.

In the first exploratory interview, to probe further into whether the PSTs understood and/or could use the properties of geometric reflection for performing reflection, we asked them how they would reflect this triangle without using GeoGebra (see Figures 4a and 4b). The following explanations of their approaches show how they thought about this task:

> PST1: We need to consider that there are reflections of points at the same distance to the reflection line.
>
> PST2: By drawing perpendicular from here [referring to point J to the reflection line]. I would have moved the same point here [referring to the other side of the reflection line] at the same distance. I will do the same process for each corner and draw the figure that way.
>
> PST3: With the help of a ruler, or a compass, for example, when I reflect this point [referring to point J] (see Figure 4b), it should be perpendicular to the reflection line. I can measure the distance of point J to the reflection line by using a ruler and I can reflect the point to the other side of the reflection line.
>
> PST4: We need to note that it [referring to the points of the triangle] should be perpendicular to the reflection line. I think that we will reflect the taken points [referring to the points of the triangle] vertically to the reflection line so that the opposite figure should be the same distance to the reflection line.

These explanations indicated that while PST1 considered only the equidistance property, the other three PSTs considered both the equidistance and perpendicularity properties for performing geometric reflection. Also, when they reflected the triangle, all the PSTs considered the points of the figure rather than the figure as a whole, which is evidence that they all understood the relationship between the pre-image and image points and the reflection line. During the all three interviews, there was no clear evidence that PST1 knew about or used the perpendicularity property for performing a geometric reflection. We inferred from PSTs' explanations that use of GeoGebra entailed limitations for PSTs' understanding of the role of the reflection line (use of the properties of equidistance and perpendicularity). Because the reflect-about-line tool directly applies geometric reflection properties (e.g., equidistance and perpendicularity) to the figure, the PSTs did not have to think about these properties when they performed a geometric reflection using GeoGebra.

The findings of the study indicated that, after they completed all the activities during the first exploratory interview, PTS2, PTS3, and PTS4 exhibited a mapping view in their understanding of the reflection line. Only PTS1 showed no change in her understanding of the concept of the reflection line and therefore retained a motion view for this concept. The PSTs' development in transition from motion view to mapping view for the reflection line is shown in Figure 5.

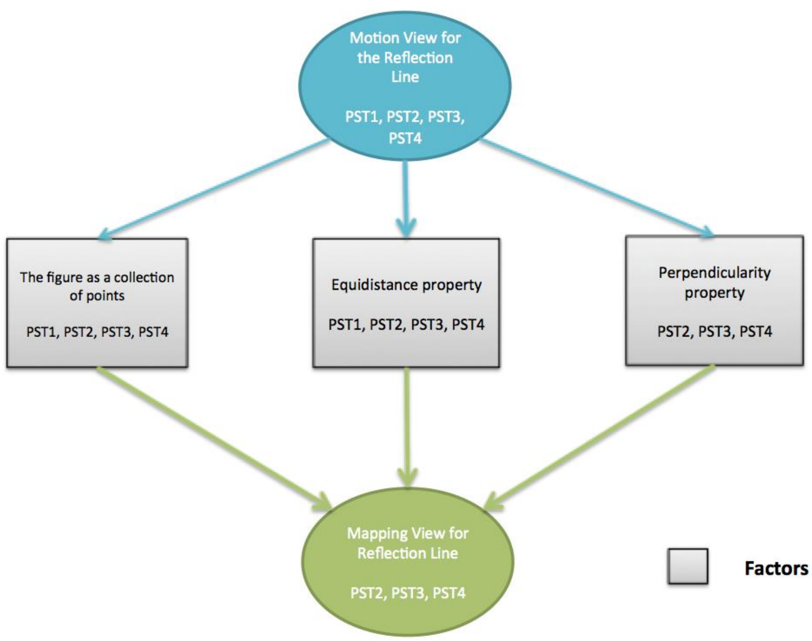


Figure 5. PSTs' transition from motion to mapping view in their understanding of reflection line

4.2. PSTs' understanding of the domain

In the first exploratory interview, all four PSTs demonstrated a motion view of the domain concept of geometric reflection by considering only given points or figures (e.g., edge points, perimeter of the figure) rather than all points in the plane when performing a geometric reflection. For instance, when asked to perform a reflection using GeoGebra to draw the image of a triangle over an oblique reflection line that did not intersect the triangle (see Figure 6), PST1 first labeled the three edges of the triangle (H, I, J) and used the reflect-about-line tool to reflect three edge points, and then used the segment tool to connect segments (e.g., H`J`, J`I`, H`I`).

Figure 6. PST1's drawing of a reflection on an oblique line

When PST1 was asked to explain what she reflected, she elaborated her ideas in the following excerpt:

PST1: I reflected points.

R : Which points did you reflect?

PST1: Edge points [referring to H, I, J]

R : Are there any other points being reflected beside edge points?

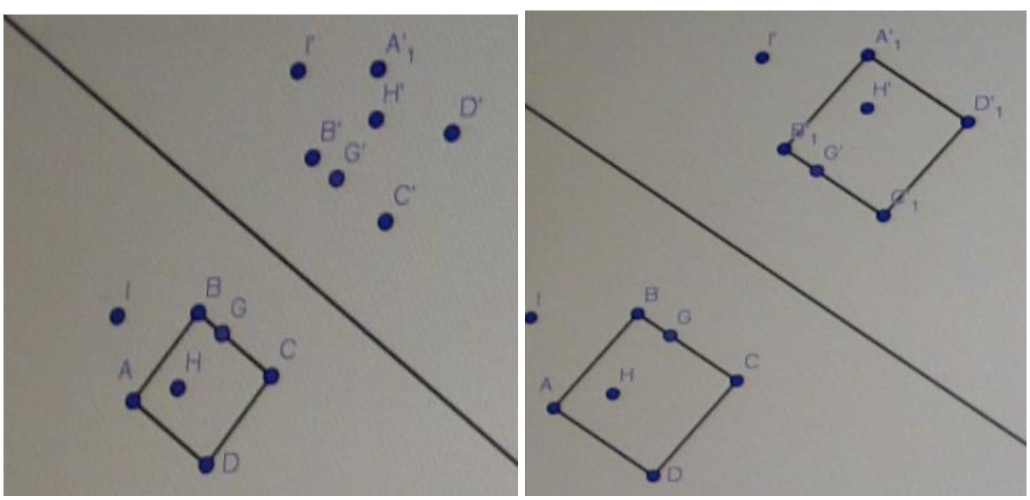
PST1: Actually, I reflected these lines [referring to HJ, JI, HI] since these lines consist of infinitely many points, I reflected all these points [referring to infinitely many points on the segments HI, JI and HJ]. But, I only reflected edge points. Due to the convenience of the GeoGebra program, the other lines were directly reflected in the program.

R : Okay, are there any other points being reflected beside the edge points and lines?

PST1: No, I did not reflect.

We inferred from her explanation that she considered only the edge points of the triangle and perimeter of the triangle while performing a geometric reflection. Even though she knew the triangle consists of infinitely many points on the perimeter, she did not consider any points inside and outside of the triangle which means she had a motion view of the domain while performing the geometric reflection.

To prompt PSTs to consider all points in the plane when performing a geometric reflection, in the first exploratory interview we also posed a task that included points both interior and exterior to the figure (see Figure 7a). All four PSTs selected points (A, B, C, D, G, H, I) and used the reflect-about-line tool to reflect all seven points and then used the segment tool to connect segments (see Figure 7b). After completing the reflection, they were asked what they reflected. All PSTs claimed that they reflected all labeled points and the perimeter of the figure. When asked whether other points beside labeled points and segments were being reflected, all stated "no." Thus, being given both interior and exterior points to provoke understanding of the domain as comprising all points in the plane did not lead the PSTs to consider unlabeled points, indicating they might not know the definition of the plane or be able to use it in performing geometric reflection. This observation provided evidence that they had a motion view of the domain.



(7a) (7b)

Figures 7a and 7b. PST4's drawing a reflection on an oblique line.

To further investigate PSTs' understanding of the plane, they were asked to define it. They explained the concept as follows:

PST1: The universe where there are a lot of objects that we can work on… It is infinite… It covers everything like these lines [referring to the highlighted lines] and the points.

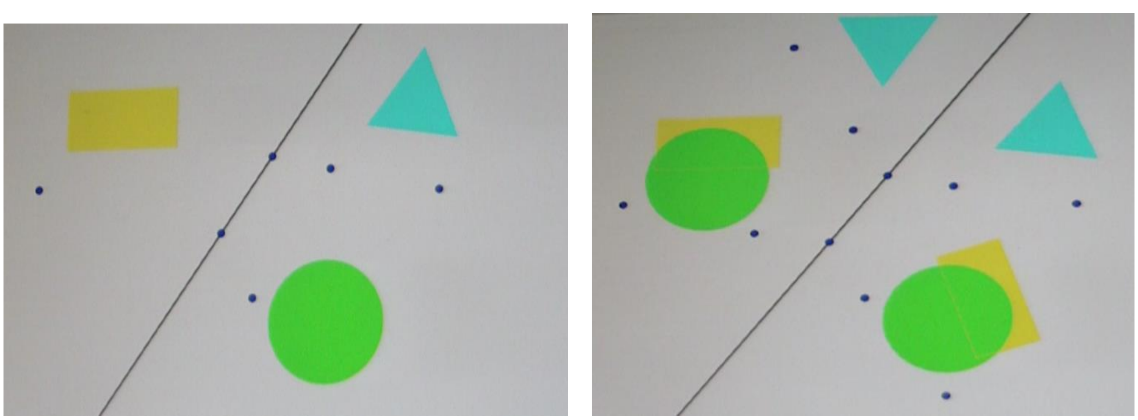
PST2: The plane is the largest all-encompassing set… The plane is composed of a set of points.

PST3: We can call it the set we are working on in the plane… There are points on it.

PST4: The plane consists of the locations of the points and points themselves.

Based on these responses, we inferred that all the PSTs knew the definition of the plane. However, they did not apply it when performing the geometric reflection, perhaps because of the convenience of the GeoGebra program, which automatically reflects what is selected on the screen, so the PSTs considered only given points or figures in their understanding of the domain of geometric reflection.

During the first exploratory interview, the PSTs were asked to reflect the points and figures from half of the plane (pre-image plane) to the other half of the plane (image plane) although in geometric reflection infinitely many points are reflected from the pre-image to the image plane and vice versa. In the second and third exploratory interviews, to prompt the PSTs to consider interior points and both halves of the plane, we asked them to reflect a rectangle with a yellow interior, a triangle with a blue interior, a circle with a green interior, and some points outside of the figures for both planes (see Figure 8a). All PSTs easily used the GeoGebra tools (e.g., reflect-about-line, segment tools) to reflect all figures and points for both planes (see Figure 8b). After performing the reflection, they all responded to the question of what they reflected by stating that they reflected the labeled points and the circular, rectangular, and triangular areas.



(8a) (8b)

Figures 8a and 8b. PST2's drawing of geometric reflection for both sides of the plane

R: Can you explain in detail how you performed the reflection?

PST3: I reflected the rectangle area and a point from this plane [referring to the left side of the plane] to another plane [referring to the right side of the plane]. Also, I reflected three points and a triangle area and circle area from this plane [referring to right side of the plane] to another plane [referring to the left side of the plane]. These two points reflected themselves since their distance to the reflection line is zero.

R: Are there any other points being reflected beside the points and figures?

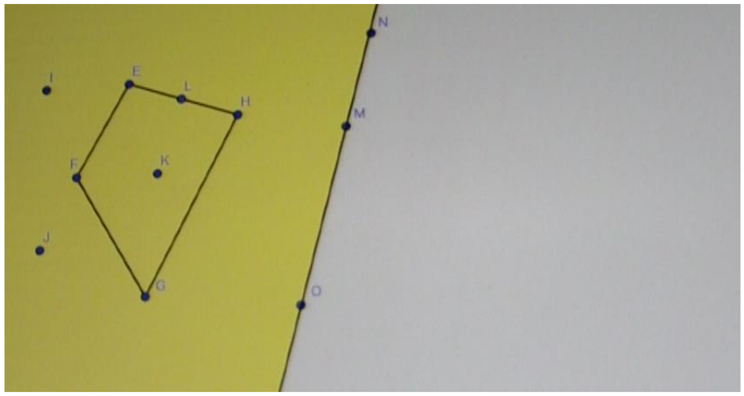
PST3: No, I did not reflect [any more points].

R: So, when you perform a reflection, is there any difference between this task [referring to figure 8b] and the previous task [referring to figure 7]?

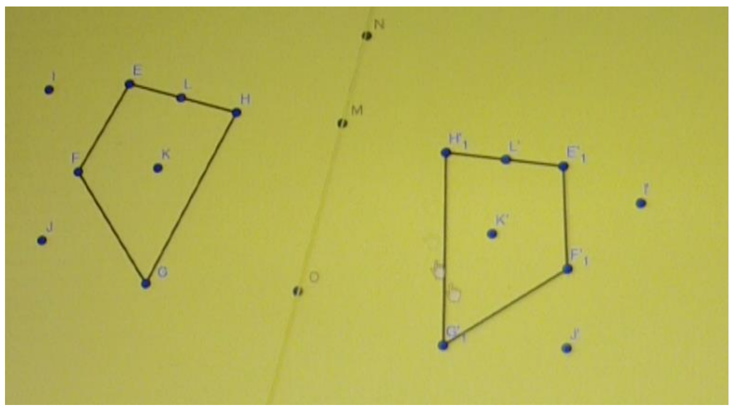
PST3: This one [referring to figure 8b] was filled, the other [referring to figure 7a] was empty, had only corners and perimeter. We could not call it [referring to figure 7a] a quadrilateral area. We can find an area for this [referring to figure 8b].

PST3's explanation suggests that he considered only the visible points and figures with shaded areas for both planes in performing the geometric reflection, from which we inferred that coloring inside the figures or labeled points outside of the figures did not lead him to consider more points in the plane when performing the geometric reflection. Also when he was asked to compare uncolored figures (Figure 7a) and colored figures (Figure 8b), he considered the former "empty" and the latter "filled," indicating that he still had a motion view of domain in geometric reflection as he did not consider reflecting all points in both the pre-image and the image planes.

In the third exploratory interview, to encourage the PSTs to consider all points in the pre-image half of the plane, they were asked to reflect a quadrilateral with shaded yellow interior and outside color, and some points inside and outside of the figures (see Figure 9a). All PSTs easily used the properties of GeoGebra (e.g., reflect-about-line, segment tools) to reflect the quadrilateral, points and the yellow area (see Figure 9b). After performing the reflection, they were asked what they reflected. All PSTs indicated that they reflected labeled points, quadrilateral, inside and outside of the quadrilateral area.



(9a)



(9b)

Figure 9a and 9b. PST3's drawing a reflection on an oblique line

To unpack PST3's understanding for domain, we asked him about his work:

R : What did you reflect?

PST3: I reflected the four sides of the rectangle. So I reflected four line segments and points connected end-to-end. I also reflected the yellow area.

R: What do you mean by the yellow area?

PST3: This is the shaded area behind it ["it" refers to the quadrilateral]. We can call it a set or plane.

R: Did you reflect anything other than these? [these refers to the left half of the plane]

PST3: No, I did not reflect.

R: What was on the right of the plane before you applied reflection?

PST3: There was nothing.

We inferred from PST3's explanation that he still considered only visible points and given figures with shaded areas when performing geometric reflection and that the coloring inside and outside the figures and labeling of some points inside and outside of the figure did not lead him to consider all points in the plane when performing the geometric reflection. Also, when asked what was on the right of the plane before he applied reflection, he replied that it was empty. The other PSTs provided similar responses. At the end of the third interview, all PSTs still conceived the domain as a single figure for performing geometric reflection demonstrating a motion view of the domain for geometric reflection. This persistence could be related to their definitions of the plane, which were limited and inconsistent when they performed a geometric reflection. Figure 10 shows PSTs' understanding of the domain throughout the first, second, and third interviews.

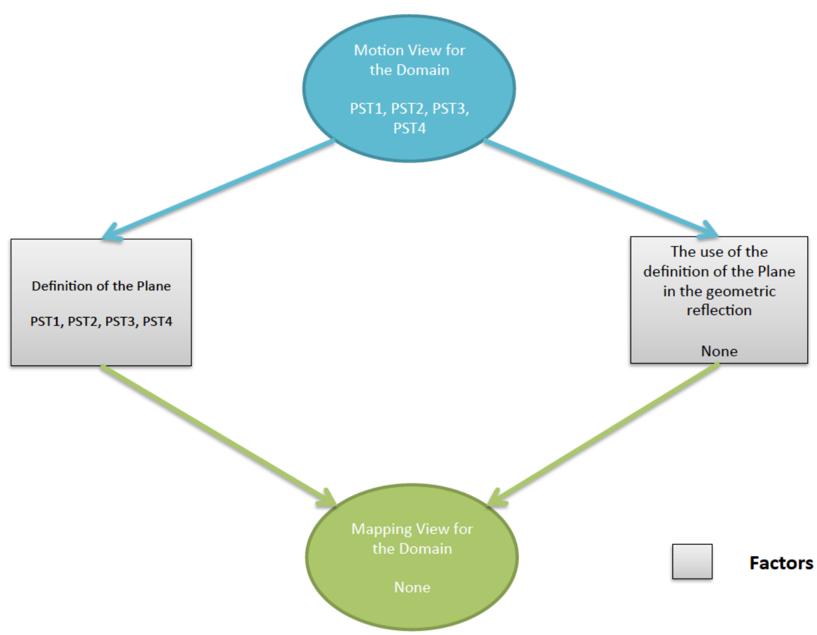


Figure 10. PSTs' understanding of domain in transition from motion to mapping view

4.3. PSTs' understanding of plane

The analyses of the all exploratory interviews showed that all PSTs considered the plane in geometric reflection as empty. When they were given colored and uncolored figures, they described figures that were not colored inside as empty. For instance, when asked to apply geometric reflection to an image showing an uncolored circle and inside a colored circle, they considered the two circles as substantively different from each other. PST3 explained, "one of the circles is colored and the other is empty." PST2 further asserted, "while we cannot calculate the area of an empty circle, we can calculate the area of a colored circle." From such remarks we inferred that they considered the plane as empty rather comprising infinitely many points.

During the first and second exploratory interviews, it was unclear whether PSTs considered the points or figures as moveable on the plane or as subsets of the plane. All PSTs used "reflected" as a word to describe their action when they performed geometric reflection using GeoGebra. In the third interview, they were asked, "When you perform a reflection, is there any movement of the points or figures from one half of the plane to another half of the plane?" They answered as follows:

> PST1: We just apply reflection to the points or figures. There is no movement. We do not move the points or figures, only the location of the coordinates changes.
>
> PST2: Yes, there is movement. It has to be of equal length to the reflection line. We can get the same figure when we move and rotate it.
>
> PST3: Actually, yes, we are moving a point twice as much as the distance here [referring to the combined distances of a point in the left half of the plane to the reflection line and of a reflected point in the right half to the reflection line], we are moving it.
>
> PST4: The position and direction of the figure has changed. So, the figure came from here to here, there is actually movement.

We inferred from these explanations that only PST1 considered there was no movement while the others considered the points or figures as separable from rather than a subset of the plane. Their use of variations of the word "move" clearly indicated that these three PSTs still had a motion view of the plane for geometric reflection. On the other hand, even though PST1 considered the points or figures as a part of the plane, she considered the plane as empty, indicating that her definition of the plane was not consistent with her mathematical understanding of the relationship between the figures or points and the plane. Therefore, PST1 also had a motion view of the plane. Figure 11 shows PSTs' understanding of plane throughout the first, second, and third interviews.

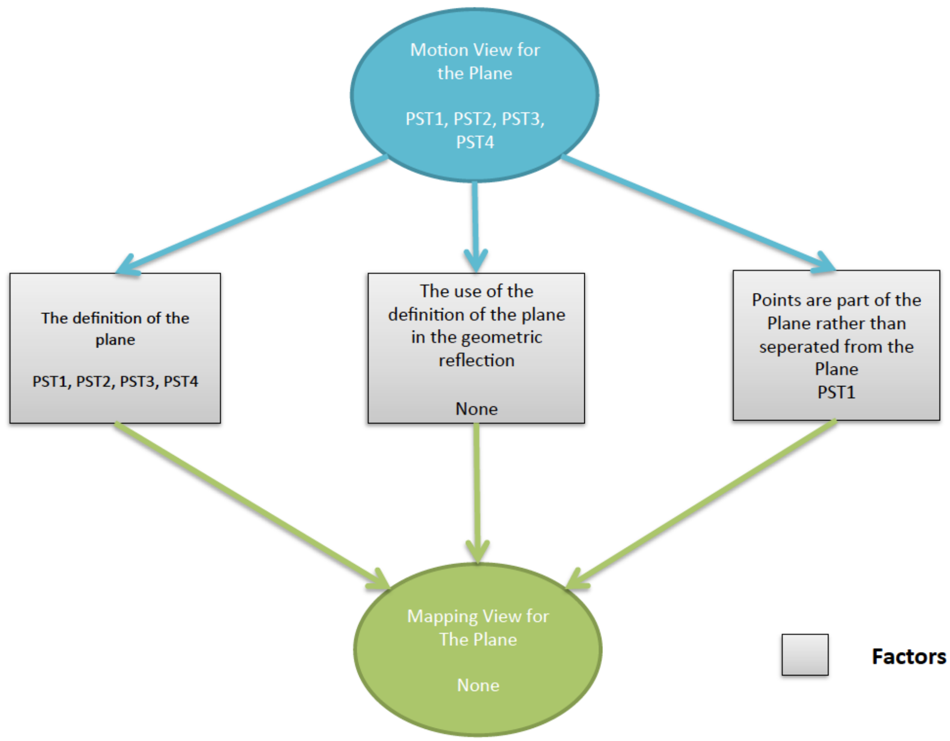


Figure 11. PSTs' understanding of plane in transition from a motion to a mapping view

4.4. PSTs' final status regarding their understanding in the sub-concepts of geometric reflection

The findings of the study revealed that using GeoGebra obstructed the PSTs' transition from a motion view to a mapping view of geometric reflection, particularly with regard to their understanding of the sub-concepts of the reflection line, domain and plane. In other words, the use of GeoGebra promoted a motion view rather than a mapping view of geometric reflection. Table 4 summarizes the PSTs' final mental structures of their understanding of reflection line, domain, and plane in geometric reflection after they had completed all the geometric reflection tasks using GeoGebra.

Table 2. PSTs' final status regarding their understanding of reflection line, domain, and plane in geometric reflection

	Geometric Understanding of the Reflection Line	**Geometric Understanding of the Domain**	**Geometric Understanding of the Plane**
PST1	Motion	Motion	Motion
PST2	Mapping	Motion	Motion
PST3	Mapping	Motion	Motion
PST4	Mapping	Motion	Motion

As shown in Table 2, it was observed that, at the end of the clinical interviews, while all PSTs except for PST1 had a mapping view of the reflection line, they all retained motion views of domain and plane.

5. Discussion

5.1. PSTs' understanding of reflection line

The findings gathered from the first interview indicated that initially all PSTs had a motion view of the reflection line in their understanding of geometric reflection. Also their initial definitions of geometric reflection and their work on the interview tasks using GeoGebra indicated they had difficulty in referring to or using the properties of equidistance and perpendicularity (Flanagan, 2001; Yanik, 2013). A possible explanation for this difficulty might be that GeoGebra's reflect-about-line tool automatically provides equidistance and perpendicularity, absolving the PSTs of any need to consider these properties when performing geometric reflection. On the other hand, Hollebrands (2007) and Yanik (2013) reported that such affordances as GeoGebra's dragging and measurement features were helpful for identifying the properties of geometric transformations. In our study, however, none of the PSTs used the dragging and measurement features of the program for performing geometric reflection. When asked about how to perform reflection if they were working in paper-pencil environment, PST1 mentioned only the equidistance property, while the other three PSTs said that they would use both the equidistance and perpendicularity properties for performing geometric reflection. These responses suggest that while they knew about the equidistance and perpendicularity properties for performing geometric reflection in a non-technological environment, they did not use this knowledge when GeoGebra performed the related operations for them.

In their initial interviews, PST2 and PST4 provided definitions that implied that they considered the reflection of a figure as a whole entity rather than as a collection of points or segments. However, when they were given triangle or rectangle reflection tasks, they first thought about the corner points of these figures, suggesting they could consider the figure as a collection of points, as they would in a paper-pencil environment (Akarsu, 2018; Yanik, 2006). However, after they used the GeoGebra reflect-about-line and segment tools, the PSTs' explanations suggested a whole figure view, even though none of them used the GeoGebra polygon tool, which directly reflects the figure as a whole. Studies focusing on the role of GeoGebra dominantly advocate its positive effects on the learning and teaching of mathematical concepts (Yanik, 2013; Zulnaidi et al., 2020) and subjects in mathematics (e.g., Öçal, 2017). However, in this study, GeoGebra did not help the PSTs develop a mapping view of the reflection line, while the researchers' prompting questions and references to working in a paper-pencil environment did help. These findings were compatible with those of some other studies (e.g., Flanagan, 2001; Yanik, 2006).

5.2. PSTs' understanding of domain

Various researchers have asserted that to have a mapping view of the domain in geometric reflection, learners should correctly define the plane and use this definition in performing the reflection (Hollebrands, 2003; Yanik, 2006). These researchers have also emphasized that even when participants provided correct formal or informal definitions of the plane, initially they could not appropriately use the concept in their geometric reflections. In their initial interviews in this study, all the PSTs could either provide the correct definition of the plane concept or show evidence that they had this knowledge. Throughout the task based interviews, however, they reverted to reflecting only the highlighted figures, labeled points and line segments, and/or colored areas and ignored all non-highlighted points in the plane. Therefore, at the end of the interviews, they had consistently exhibited a motion view of the domain in the geometric reflection. This situation can be related to the influence of GeoGebra on their understanding of the domain concept.

The tasks given in the interviews were designed to stimulate the PSTs' consideration of using all points in their reflections, and the researchers' questions and follow-up tasks were intended to guide their understanding. For example, after the PSTs had reflected only a triangle's points, they were provided with a figure with labeled points both inside and outside the geometric shape. At the end of the final interview, the whole left side of the plane was colored to prompt reflecting all points. Moreover, after the end of each task given, the researchers asked the PSTs whether there were any other points/figures left to reflect. However, none of the PSTs showed or implied understanding that there were infinitely many points in the pre-image plane and these points were also reflected. This lack of understanding could be due to their consistent reliance on the reflect-about-line and segment tools, which led them to focus on reflecting the labeled points and figures of the pre-images while feeling no necessity to apply their definition of the plane to the geometric reflection because GeoGebra was doing all the work for them. Similar to the findings regarding PSTs' understanding of the reflection line, GeoGebra preempted their own reasoning and kept them from applying a mapping view to the concept of domain in geometric reflection. For example, based on their definitions, the PSTs could be expected to understand that the entire plane provided was not empty but comprised infinitely many points both inside and outside of the focal figures. In previous studies, participants working in paper-pencil environments (e.g., Akarsu, 2018; Yanik; 2006) investigated each component of the provided pre-images in reflection tasks. In such an environment, they chose arbitrary points from plane's regions [shaded areas] and reflected them, and prompted by the researchers' questioning similar to that in this study, they developed the understanding that they were mapping all points in the pre-image plane and that the provided figures and points were subsets of this plane. However, in this study, PSTs did not need to do such in-depth investigation of points, figures and shaded areas, because GeoGebra and its reflection tools did all the work automatically. Similar to these findings, Hollebrands (2003) pointed out that students needed only to select certain points, line segments, or other figures from the pre-image plane when performing geometric reflections in DGE. Accordingly, the PSTs in this study might have assumed that the reflection applied only to the selected objects in the plane and interpreted the domain as composed only of the points forming these objects. In short, although they knew the definition of the plane, the PSTs could not apply it.

Secondly, DGE, including GeoGebra, presents the end product of the geometric reflection, generally concealing the process involved. When provided with a shaded area to be reflected, for example, the PSTs in this study focused only on this planar region and did not think about the points which composed it, which can be related to the nature of GeoGebra as a presentational rather than a problem-solving program. In previous studies (Flanagan, 2001; Yanik, 2006), however, the participants dealt with points composing the shaded region, a step that could be interpreted as midway between the motion view (thinking only about the labeled points in the reflection) and the mapping view (thinking that all labeled and unlabeled points in the plane are reflected). After participants in such studies could understand that any given planar region was composed of infinitely many points, they could develop an understanding that the plane with which they were working was also composed of infinitely many points, all of which were reflected to the other side of the reflection line. In this study, however, GeoGebra obscured this crucial step in the PSTs' reasoning by directly leading them to consider the shaded area for the intended reflection as a whole rather than as a collection of infinitely many points. Therefore, none of them could progress to a mapping view of the domain concept in the geometric reflection.

5.3. PSTs' understanding of plane

As discussed in relation to their understanding of the domain concept, the PSTs knew the definition of a plane but could not apply it to performing a geometric reflection. The findings from the clinical interviews showed that the PSTs used the reflect-about-line tool to select colored areas to perform reflections. However, they did not think about the meaning of this use of the tool, which was that they were choosing all the points composing the selected area. That is, using the reflect-about-line tool encouraged the perception that they were choosing a single figure instead of the infinitely many points composing the figure. Thus, the use of GeoGebra's reflect-about-line and segment tools promoted a motion view of the plane concept. In a paper-pencil environment, however, Akarsu (2018) found that participants had to choose some points from the colored or shaded areas to perform a reflection, which raised their awareness that there were many other points in the selected figure and in the plane itself. Instead, the PSTs in this study considered the plane empty and did not engage in this kind of investigation during their geometric reflections.

The PSTs sometimes used the dragging property of GeoGebra while performing their geometric reflection. In the third exploratory interview, they were asked whether there was any movement of the reflected figures. Three PSTs thought that a figure was moving when they were dragging the pre-image or reflection line, implying that they considered the reflected figure as separate from the plane rather than a subset of it. Only PST1 did not mention the movement of a figure from the plane. However, it was unclear in the interview whether she thought about the mapping of each point in the plane with those in the focal figures. Flanagan (2001) and Yanik and Flores (2009) similarly found that participants using Geometer's Sketchpad in a DGE environment also considered the plane on which they were working as an empty background for the focal figures. It may be inferred that the dragging property of DGE environments can distort users' understanding of what the plane really is and how the mapping of all points in the pre-image plane occurs in geometric reflections. Therefore, the GeoGebra could not help PSTs to unpack their understanding of the concept of the plane concept, but rather promoted their motion view of it.

6. Conclusions and Implications

The findings of the study demonstrate that use of GeoGebra supported a motion rather than a mapping view of geometric reflection in several ways. First, the reflect-about-line and segment tools in GeoGebra do not require PSTs to use the properties of equidistance and perpendicularity properties in their reasoning as the program automatically does this for them. Second, the use of the polygon tool supports considering a focal figure as a whole entity rather than as a collection of points. Third, when using the reflect-about-line tool to perform a reflection, it is possible that the users think of the line of reflection as a tool itself rather than a geometrical object. Fourth, the use of GeoGebra supports users' understanding of the domain as a single figure as they need to select particular objects (e.g., points, sides etc.) from the plane to apply the reflection. Using technology to select a particular object from the domain encouraged the PSTs in this study to think of reflection as discretely applied to that particular object. Fifth, the use of the dragging tool encouraged the PSTs to think that they were moving the points on the plane rather than reflecting them as a part of the plane. Therefore, using GeoGebra to perform the tasks in this study, rather than guiding the PSTs to the mapping view of geometric reflection, reinforced the motion view.

In conclusion, researchers suggest that use of DGS promotes' learners' understanding of geometric reflection (Hollebrands, 2007; Yanik, 2013). This study however demonstrates that DGS has limitations in that it supports a motion view rather than a mapping view of geometric reflection. This outcome suggests that much more investigation is needed into the role of DGS tools in helping PSTs develop a mapping view of geometric reflection and, more broadly, understand and learn how to teach specific concepts such as geometric reflection.

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